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Published in:
Nuclear Physics B

DOI:
10.1016/0550-3213(86)90503-1

Nuclear Physics B264 (1986) 653-686

SUPERCONFORMAL TENSOR CALCULUS AND MATTER COUPLINGS IN SIX DIMENSIONS

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Received 5 September 1985

Using superconformal tensor calculus we construct general interactions of $N=2$, $d=6$ supergravity with a tensor multiplet and a number of scalar, vector and linear multiplets. We start from the superconformal algebra which we realize on a 40+40 Weyl multiplet and on several matter multiplets. A special role is played by the tensor multiplet, which cannot be treated as an ordinary matter multiplet, but leads to a second 40+40 version of the Weyl multiplet. We also obtain a 48+48 off-shell formulation of Poincaré supergravity coupled to a tensor multiplet.

1. Introduction

$N=2$, $d=6$ Poincaré supergravity coupled to matter has been constructed in ref. [1]. Subsequently, it was found that [2] the theory compactifies on the product of four-dimensional Minkowski space-time and a 2-sphere, yielding chiral fermions in $d=4$. Moreover, it has been shown that [3] due to the Green-Schwarz mechanism [4], the theory is anomaly-free for a special choice of the gauge group (in particular $E_6 \times E_7 \times U(1)$) and particle content. Monopole compactification of the theory yields two families of chiral fermions with no residual supersymmetry, and hence no massless scalars. If massless scalars had arisen in $d=4$, they could be used in breaking the Yang-Mills symmetries. A large number of other matter couplings to $N=2$, $d=6$ supergravity are also anomaly-free [5].

Clearly it is desirable to extend, if possible, the results of ref. [1], so as to find a more realistic compactification scheme where, firstly the correct number of families of chiral fermions would emerge, and secondly an $N=1$ supersymmetry would remain in $d=4$. The latter is desired, in particular, to generate massless scalars in $d=4$.

It is well-known that local conformal supersymmetry [6] is a very convenient framework to study general matter couplings to supergravity theories. By imposing

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suitable gauge conditions (breaking superconformal symmetries down to super-Poincaré symmetries) couplings of well-chosen matter multiplets to the Weyl multiplet give rise to off-shell formulations of Poincaré supergravity [7] (reviews of this method are given in [8, 9]). This has been used to construct general matter couplings to $N=1$ [10], $N=2$ [11, 12] and $N=4$ [13] supergravities in $d=4$. In this paper we perform the first steps in the superconformal program in six dimensions. We will obtain multiplets very similar to those of refs. [11, 12]. We first construct the superconformal generator algebra, which is OSp(6, 2/1). The generators are labeled

$$T_A = M_{ab},\ P_a,\ K_a,\ D,\ Q_{\alpha i},\ S_{\alpha i},\ U_{ij}\,, \tag{1.1}$$

where $a, b, \ldots$ are Lorentz indices, α is a spinor index and $i, j, \ldots = 1, 2$. M_{ab}, P_a are the usual Poincaré generators, K_a is the boost, D the dilatation, $Q_{\alpha i}$ is the doublet of usual supersymmetry generators, while $S_{\alpha i}$ are those of the special supersymmetry. U_{ij} is symmetric and labels the triplet of SU(2) generators, satisfying $(U_{ij})^* = U^{ij} \equiv \varepsilon^{ik}\varepsilon^{jl}U_{kl}$ (see appendix A for notations). We next realise this algebra on the following set of fields

$$e_\mu^a,\ \psi^i_\mu,\ V_\mu^{ij},\ b_\mu,\ T^-_{abc},\ \chi^i,\ D\,, \tag{1.2}$$

where the first four are the gauge fields corresponding to the generators P_a, $Q_{\alpha i}$, U_{ij} and D. The anti-selfdual tensor T^-_{abc}, the spinor χ^i and the scalar D are matter fields*. As will be explained in the text they are needed in order to obtain a massive spin-2 representation of the $d=6$ super-Poincaré algebra, which has 40 Bose plus 40 Fermi degrees of freedom. The field b_μ can be gauged to zero by using the conformal boosts. We then impose the superconformal commutator algebra on the components of the following matter multiplets [14]: (a) Yang–Mills multiplet, (b) hypermultiplet (scalar multiplet), (c) tensor multiplet, (d) linear multiplet and (e) non-linear multiplet. The contents of these multiplets are listed in table 5 in the text. For the hypermultiplet this can only be done modulo constraints which will become field equations. The linear multiplet is an off-shell version of this multiplet at the expense of introducing an antisymmetric tensor. The other off-shell version, the relaxed hypermultipet [14], is not treated here. Also for the tensor multiplet we need constraints in order to impose the superconformal algebra. In this case, however, these constraints cannot be interpreted as field equations. Instead two of them can be solved for the fields D and χ in (1.2) while the third becomes a Bianchi identity. The multiplet contains a selfdual field strength F^+_{abc} which is in fact the obstacle for constructing a Lorentz invariant action for this multiplet separately and thus for considering this multiplet as an ordinary matter multiplet [15]. By this constraint it can be interpreted as the selfdual part of the field strength of an antisymmetric tensor $B_{\mu\nu}$, while T^-_{abc} in (1.2) becomes the antiselfdual part. This combined Weyl-tensor multiplet will become the basis for building actions. We need

* All bosonic fields are real if their Lorentz and world indices are different from 6. For $\mu \neq 6$ $(V_{\mu ij})^* = +(V_\mu)^{ij}$.

the action of a compensating scalar or linear multiplet to obtain Poincaré supergravity coupled to a tensor multiplet. This basic system can be coupled to actions for scalar multiplets with or without gauge interactions, kinetic terms for the Yang-Mills multiplets and linear multiplets.

Using the compensating linear multiplet we obtain also a 48+48 off-shell multiplet which describes Poincaré supergravity coupled to a tensor multiplet. This auxiliary field formulation is of the type of the new minimal formulation of $N=1$ supergravity [16] as it has an auxiliary gauge antisymmetric tensor and a SO(2) invariance gauged by an auxiliary vector.

Sect. 2 contains a detailed description of the construction of the $N=2$, $d=6$ Weyl multiplets. In sect. 3 we construct the full superconformal transformations of the matter multiplets. Actions are discussed in sect. 4 and a recapitulation with conclusions is presented in sect. 5. Our conventions are given in appendix A while appendix B contains the $N=2$, $d=6$ superconformal generator algebra.

2. The $N=2$, $d=6$ Weyl multiplet

As a first step in the construction of the $N=2$, $d=6$ Weyl multiplet we consider the superconformal algebra in six dimensions which is OSp(6, 2/1). The nonzero commutators between the different generators of this algebra are given in appendix B. For our conventions we refer to appendix A. To each generator T_A of the superconformal algebra we assign a gauge field h_μ^A in the following way:

$$h_\mu^A T_A = e_\mu^a P_a + \omega_\mu^{ab} M_{ab} + b_\mu D + f_\mu^a K_a + \bar{\psi}^i_\mu Q_i + \bar{\phi}^i_\mu S_i + V_\mu^{ij} U_{ij}\,. \tag{2.1}$$

Here ψ^i_μ and ϕ^i_μ are SU(2) Majorana-Weyl spinors of positive and negative chirality respectively:

$$\psi^i_\mu = +\gamma_7 \psi^i_\mu\,, \qquad \phi^i_\mu = -\gamma_7 \phi^i_\mu\,. \tag{2.2}$$

Furthermore the gauge field V_μ^{ij} satisfies

$$V_\mu^{ij} = V_\mu^{ji}\,, \qquad V_\mu{}^i{}_i = 0\,, \qquad (V_{\mu ij})^* = V_\mu^{ij} \qquad \text{for } \mu \neq 6\,. \tag{2.3}$$

Using the structure constants of the superconformal algebra:

$$[T_A; T_B\} \equiv T_A T_B \mp T_B T_A = f_{AB}{}^C T_C \tag{2.4}$$

(+ sign if A and B are fermionic) and the basic rule

$$\delta h_\mu^A = \partial_\mu \varepsilon^A + \varepsilon^C h_\mu^B f_{BC}{}^A \tag{2.5}$$

one can immediately determine the gauge transformations of the superconformal fields defined in (2.1). We have listed these transformations below.

$$
\begin{aligned}
\delta &= \bar{\varepsilon}Q + \bar{\eta}S + \Lambda_D D + \varepsilon^{ab}M_{ab} + \Lambda^a_K K_a + \Lambda^{ij}U_{ij}\,,\\
\delta e^a_\mu &= \tfrac{1}{2}\bar{\varepsilon}\gamma^a\psi_\mu - \Lambda_D e^a_\mu - \varepsilon^{ab}e_{\mu b}\,,\\
\delta\psi^i_\mu &= \mathscr{D}_\mu\varepsilon^i + \gamma_\mu\eta^i - \tfrac{1}{2}\Lambda_D\psi^i_\mu + \tfrac{1}{2}\Lambda^i{}_j\psi^j_\mu - \tfrac{1}{4}\varepsilon^{ab}\gamma_{ab}\psi^i_\mu\,,\\
\delta b_\mu &= \partial_\mu\Lambda_D - \tfrac{1}{2}\bar{\varepsilon}\phi_\mu + \tfrac{1}{2}\bar{\eta}\psi_\mu - 2\Lambda_{K\mu}\,,\\
\delta\omega_\mu{}^{ab} &= \partial_\mu\varepsilon^{ab} + 2\omega_{\mu c}{}^{[a}\varepsilon^{b]c} - \tfrac{1}{2}\bar{\varepsilon}\gamma^{ab}\phi_\mu - \tfrac{1}{2}\bar{\eta}\gamma^{ab}\psi_\mu + 4\Lambda_K{}^{[a}e^{b]}_\mu\,,\\
\delta V^{ij}_\mu &= \partial_\mu\Lambda^{ij} + \Lambda^{(i}{}_k V^{j)k}_\mu - 4\bar{\varepsilon}^{(i}\phi^{j)}_\mu - 4\bar{\eta}^{(i}\psi^{j)}_\mu\,,\\
\delta\phi^i_\mu &= \mathscr{D}_\mu\eta^i - f^a_\mu\gamma_a\varepsilon^i + \Lambda^a_K\gamma_a\psi^i_\mu + \tfrac{1}{2}\Lambda^i{}_j\phi^j_\mu + \tfrac{1}{2}\Lambda_D\phi^i_\mu - \tfrac{1}{4}\varepsilon^{ab}\gamma_{ab}\phi^i_\mu\,,\\
\delta f^a_\mu &= \mathscr{D}_\mu\Lambda^a_K + \Lambda_D f^a_\mu - \tfrac{1}{2}\bar{\eta}\gamma^a\phi_\mu - \varepsilon^{ab}f_{\mu b}\,.
\end{aligned}
\tag{2.6}
$$

The spinors ε^i and η^i are positive and negative chiral respectively.

$$
\varepsilon^i = +\gamma_7\varepsilon^i\,, \qquad \eta^i = -\gamma_7\eta^i\,. \tag{2.7}
$$

$\mathscr{D}_\mu$ is covariant with respect to the linearly realised symmetries

$$
\begin{aligned}
\mathscr{D}_\mu\varepsilon^i &= \partial_\mu\varepsilon^i + \tfrac{1}{2}b_\mu\varepsilon^i + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\varepsilon^i - \tfrac{1}{2}V_\mu{}^i{}_j\varepsilon^j\,,\\
\mathscr{D}_\mu\eta^i &= \partial_\mu\eta^i - \tfrac{1}{2}b_\mu\eta^i + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\eta^i - \tfrac{1}{2}V_\mu{}^i{}_j\eta^j\,,\\
\mathscr{D}_\mu\Lambda^a_K &= \partial_\mu\Lambda^a_K - b_\mu\Lambda^a_K + \omega_\mu{}^{ab}\Lambda_{Kb}\,.
\end{aligned}
\tag{2.8}
$$

Note also that SU(2) indices are omitted when a northwest-southeast contraction is understood (see A.9).

The structure constants $f_{BC}{}^A$ of the superconformal algebra also enable us to calculate the curvature tensors $R_{\mu\nu}{}^A$ by means of

$$
R_{\mu\nu}{}^A = 2\partial_{[\mu}h_{\nu]}{}^A + h^C_\nu h^B_\mu f_{BC}{}^A\,. \tag{2.9}
$$

We have listed them in table 1.

TABLE 1

Curvatures of the $N=2$, $d=6$ superconformal algebra

$$
\begin{aligned}
R_{\mu\nu}{}^a(P) &= 2\partial_{[\mu}e^a_{\nu]} + 2b_{[\mu}e^a_{\nu]} + 2\omega_{[\mu}{}^{ab}e_{\nu]b} - \tfrac{1}{2}\bar{\psi}_\mu\gamma^a\psi_\nu\\
R_{\mu\nu}{}^i(Q) &= 2\mathscr{D}_{[\mu}\psi^i_{\nu]} + \underline{2\gamma_{[\mu}\phi^i_{\nu]}}\\
R_{\mu\nu}{}^{ab}(M) &= 2\partial_{[\mu}\omega_{\nu]}{}^{ab} + 2\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^b - \underline{8f_{[\mu}{}^{[a}e_{\nu]}{}^{b]}} + \bar{\psi}_{[\mu}\gamma^{ab}\phi_{\nu]}\\
R_{\mu\nu}(D) &= 2\partial_{[\mu}b_{\nu]} + \underline{4f_{[\mu}{}^a e_{\nu]a}} + \bar{\psi}_{[\mu}\phi_{\nu]}\\
R_{\mu\nu}{}^{ij}(V) &= 2\partial_{[\mu}V_{\nu]}{}^{ij} + V_{[\mu}{}^{k(i}V_{\nu]}{}^{j)}k + 8\bar{\psi}_{[\mu}{}^{(i}\phi_{\nu]}{}^{j)}\\
R_{\mu\nu}{}^i(S) &= 2\mathscr{D}_{[\mu}\phi^i_{\nu]} - 2\gamma_a f_{[\mu}{}^a\psi_{\nu]}{}^i\\
R_{\mu\nu}{}^a(K) &= 2\mathscr{D}_{[\mu}f_{\nu]}{}^a + \tfrac{1}{2}\bar{\phi}_\mu\gamma^a\phi_\nu
\end{aligned}
$$

The underlined terms in table 1 represent the terms proportional to a connection field multiplied by a sechsbein field. The covariant derivatives D_μ are again covariant with respect to D, M and SU(2). For convenience we have written them explicitly in the expressions for $R(P)$, $R(M)$ and $R(V)$.

The superconformal gauge fields defined in (2.1) describe $\underline{155+80}$ (bosonic+fermionic) off-shell field degrees of freedom. The bosonic degrees of freedom are described by the gauge fields $e_\mu^a(30)$, $\omega_\mu{}^{ab}(75)$, $V_{\mu j}^i(15)$, $f_\mu^a(30)$ and $b_\mu(5)$. The fermionic degrees of freedom are described by the gauge fields $\psi_\mu^i(40)$ and $\phi_\mu^i(40)$. The 30 degrees of freedom (d.o.f.) described by the sechsbein decompose into massive representations of the $d=6$ Poincaré group as a massive spin-2 representation (14 d.o.f.) plus lower-spin representations. *By definition* the $d=6$ Weyl multiplet is the smallest irreducible multiplet that contains the massive spin-2 representation.

To achieve a maximal irreducibility of the superconformal gauge field configuration we impose a maximal set of so-called conventional constraints [17] on the superconformal curvatures (see table 1). Inspection of the explicit form of these curvatures shows that $R(P)$, $R(M)$, $R(D)$ and $R(Q)$ contain terms proportional to a connection field, multiplied by a sechsbein field. Hence, these connections, $\omega_\mu{}^{ab}$, f_μ^a and ϕ_μ^i can be expressed in terms of the remaining gauge fields by imposing curvature constraints. For this purpose the following set of constraints suffices:

$$
\begin{aligned}
R_{\mu\nu}{}^a(P) &= 0 \qquad (90)\,,\\
R_{\mu\nu}{}^{ab}(M)e^\nu{}_b &= 0 \qquad (36)\,,\\
\gamma^\mu R_{\mu\nu}{}^i(Q) &= 0 \qquad (48)\,.
\end{aligned}
\tag{2.10}
$$

We have indicated the number of constraints between brackets. At first sight, it seems that one can also restrict $R(D)$, but in the presence of the first constraint of (2.10) one can show that $R(D)$ is no longer independent by virtue of a Bianchi identity.

The expressions for $\omega_\mu{}^{ab}$, f_μ^a and $\phi_\mu{}^i$, which follow from (2.10) are given by

$$
\begin{aligned}
\omega_\mu{}^{ab} &= 2e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} - e^{\rho[a}e^{b]\sigma}e_\mu^c\partial_\rho e_{\sigma c}\\
&\quad + \tfrac{1}{4}(2\bar\psi_\mu\gamma^{[a}\psi^{b]} + \bar\psi^a\gamma_\mu\psi^b) + 2e_\mu{}^{[a}b^{b]}\,,\\
f_\mu^a &= \tfrac{1}{8}(R'_\mu{}^a(M) - \tfrac{1}{10}e_\mu^a R'(M))\,,\\
\phi_\mu^i &= -\tfrac{1}{16}(\gamma^{ab}\gamma_\mu - \tfrac{3}{5}\gamma_\mu\gamma^{ab})R'_{ab}{}^i(Q)\,.
\end{aligned}
\tag{2.11}
$$

The notation $R'(M)$ indicates that we have omitted the f_μ^a dependent term that occurs in $R(M)$, while $R'(Q)$ indicates that we have omitted the ϕ_μ^i dependent term.

Since the constraints (2.10) are invariant under M, D, K and SU(2) transformations the corresponding transformations of $\omega_\mu{}^{ab}$, f_μ^a and ϕ_μ^i implied by the superconformal algebra (cf. (2.5) and (2.6)) remain unaffected. However, the constraints are not invariant under Q and S-supersymmetry and therefore the Q and S-transformations of $\omega_\mu{}^{ab}$, f_μ^a and ϕ_μ^i change by extra terms proportional to curvature tensors

[6]. For instance, requiring invariance of the constraint $R(P)=0$ under Q-supersymmetry leads to

$$\begin{aligned}\delta_Q R_{\mu\nu}{}^a(P) &= \delta_Q^{\text{gauge}} R_{\mu\nu}{}^a(P) + 2(\delta\omega_{[\mu}{}^{ab})^{\text{add}} e_{\nu]b} \\ &= \tfrac{1}{2}\bar{\varepsilon}\gamma^a R_{\mu\nu}(Q) + 2(\delta\omega_{[\mu}{}^{ab})^{\text{add}} e_{\nu]b} = 0\,. \end{aligned} \tag{2.12}$$

Here δ_Q^{gauge} denotes the variation of $R(P)$ according to the superconformal algebra, while the second term denotes the variation of $R(P)$ owing to the extra term $(\delta\omega_\mu{}^{ab})^{\text{add}}$ in the Q-transformation of $\omega_\mu{}^{ab}$. From (2.12) we deduce that this term is given by

$$(\delta\omega_\mu{}^{ab})^{\text{add}} = -\tfrac{1}{4}(2\bar{\varepsilon}\gamma^{[a}R_\mu{}^{b]}(Q) + \bar{\varepsilon}\gamma_\mu R^{ab}(Q))\,. \tag{2.13}$$

We should mention that the detailed form of the conventional constraints is not crucial, as long as they fully restrict the gauge fields in question.

When combined with the constraints (2.10) the Bianchi identities lead to further relations among the superconformal curvatures. As it turns out the only independent bosonic curvatures are $R(M)$ and $R(V)$. The *a priori* 225 components of $R_{\mu\nu}{}^{ab}(M)$ are restricted to $\underline{14}$ independent ones by the following algebraic and differential identities:

$$\begin{gathered}\hat{R}_{\mu\nu ab}(M) = \hat{R}_{ab\mu\nu}(M) \qquad (105)\,, \qquad \hat{R}_{[\mu\nu ab]}(M) = 0 \qquad (15)\,, \\ \hat{R}_{\mu\nu}{}^{ab}(M)e^\nu{}_b = 0 \qquad (21)\,, \qquad \hat{\mathscr{D}}_b\hat{\mathscr{D}}_{[d}\hat{R}_{ef]}{}^{ab}(M) = 0 \qquad (35)\,, \\ \hat{\mathscr{D}}^{[a}\hat{\mathscr{D}}_{[d}\hat{R}_{ef]}{}^{bc]}(M) = 0 \qquad (35)\,, \end{gathered} \tag{2.14}$$

where Λ denotes covariantization with respect to all the superconformal transformations except the general coordinate transformations, and here with respect to the new transformation, (2.13). We have indicated the number of independent constraints between brackets. The *a priori* 45 components of $R_{\mu\nu}(V)^i_j$ are restricted to $\underline{15}$ independent ones by the following Bianchi identities

$$\hat{\mathscr{D}}_{[a}\hat{R}_{bc]}{}^i{}_j(V) = 0\,. \tag{2.15}$$

The only independent fermionic curvature is $R(Q)$ because $R(S)$ satisfies

$$\hat{R}_{ab}{}^i(S) = -\tfrac{1}{2}(\hat{\not{\mathscr{D}}}\hat{R}_{ab}{}^i(Q) + \tfrac{2}{3}\gamma_{[a}\hat{\mathscr{D}}^c\hat{R}_{b]c}{}^i(Q))\,. \tag{2.16}$$

The *a priori* 120 components of $R(Q)$ are restricted to $\underline{32}$ independent ones by the following identities

$$\begin{gathered}\hat{\mathscr{D}}_{[a}\hat{R}_{bc]}{}^i(Q) - \tfrac{1}{18}\gamma_{[a}\hat{\not{\mathscr{D}}}\hat{R}_{bc]}(Q) - \tfrac{1}{9}\gamma_{[a}\gamma^d\hat{\mathscr{D}}_b\hat{R}_{c]d}(Q) = 0 \qquad (80)\,, \\ \gamma^{ab}\hat{R}_{ab}{}^i(Q) = 0 \qquad (8)\,. \end{gathered} \tag{2.17}$$

Although we have now achieved a maximal irreducibility of the superconformal gauge field configuration the above procedure does not guarantee that these gauge fields constitute a complete field representation of the superconformal algebra. The following counting argument shows that this is indeed not the case. The constraints (2.10) eliminate $\underline{126+48}$ (bosonic+fermionic) field degrees of freedom. Hence after imposing these constraints we are left with $\underline{(14+15)+32}$ field degrees of freedom. The bosonic degrees of freedom are described by the independent gauge fields $e_\mu^a(14)$ and $V_{\mu j}^i(15)$, while the fermionic degrees of freedom are described by $\psi_\mu^i(32)$. These gauge field degrees of freedom form one massive spin-2, two massive spin-$\frac{3}{2}$ and three massive spin-1 representations of the $d=6$ Poincaré algebra. In table 2 we have indicated these representations together with the spins, which are contained in a $N=2$ massive spin-2 representation of the $d=6$ super-Poincaré algebra. From this table we immediately see that additional matter fields [11] must be added to the gauge fields in order to obtain such a massive spin-2 representation. We have indicated these matter fields in the fourth column of table 2. Here T_{abc}^- is an antisymmetric tensor of negative duality (i.e. $T_{abc}^- = -\frac{1}{6}i\varepsilon_{abcdef}T_{def}^-$), χ^i is an SU(2)-Majorana Weyl spinor of negative chirality (i.e. $\chi^i = -\gamma_7\chi^i$) and D is a real scalar. The duality of T_{abc}^- and the chirality of χ^i are fixed by supersymmetry. Note that instead of T_{abc}^- we could also use either a second-rank antisymmetric tensor field $B_{\mu\nu}$ or two vector fields. The first possibility will be considered in subsect. 3.3 (in this case the spinor and the scalar have different properties and hence will be denoted by ψ^i and σ respectively). The second possibility leads to inconsistencies when these vector fields are coupled to the SU(2) gauge fields V_μ^{ij} [18].

Starting from the linear transformation rules of the superconformal gauge fields given in (2.6) and the matter fields T_{abc}^-, χ^i and D introduced above we can now construct the full nonlinear $N=2$, $d=6$ Weyl multiplet by applying an iterative procedure which we describe below. Since this procedure can be applied in different situations as well we will first describe it in a more general context and then focus on the $d=6$ case. The method consists of the following steps:

(i) First write down the *linearized* Q-transformation rules (i.e. only containing terms linear in the fields) for both the gauge fields and matter fields. The terms

TABLE 2

Massive spin-2 representation of the $d=6$ super-Poincaré algebra

Spin s	$N=2$	Gauge fields	Matter fields		
2 (14)	1	1 e_μ^a			
$\frac{3}{2}$ (8)	4	4 ψ_μ^i			
1 (5)	5	3 $V_{\mu j}^i$	2 T_{abc}^-		$B_{\mu\nu}$
$\frac{1}{2}$ (2)	4		4 χ^i	or	ψ^i
0 (1)	1		1 D		σ

The numbers between brackets in the first column represent the dimensionality of each spin. Those in the second column denote the number of spins contained in a massive spin-2 representation. The numbers in the third and fourth columns indicate the massive spin states, which are described by the superconformal gauge fields and the added matter fields, respectively.

proportional to the gauge fields follow from the superconformal algebra (cf. (2.6)). The ones proportional to the matter fields can be found as follows. One first writes down the most general ansatz containing a number of constant coefficients. If derivatives of gauge fields are needed one uses the corresponding superconformal curvature tensor (cf. table 1). One now fixes the coefficients in the transformation rules by requiring that the commutator of two supersymmetry transformations with parameters ε_1 and ε_2 respectively gives a translation, e.g. in $d=6$:

$$[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)] = \tfrac{1}{2}\bar{\varepsilon}_2\gamma^\mu\varepsilon_1\partial_\mu\,. \tag{2.18}$$

The explicit form of the parameter on the r.h.s. of (2.18) follows immediately from the $\{Q, Q\}$ anticommutator given in (B.3) by identifying $\delta_Q(\varepsilon)$ with $\bar{\varepsilon}Q = \varepsilon^{\alpha i}Q_{\alpha i}$.

(ii) Determine the bosonic transformation rules of the matter fields. The Lorentz transformations follow immediately from the index structure of these fields. Their D transformations are characterized by assigning them a Weyl weight w in the following way:

$$\delta_D\phi \equiv w\Lambda_D\phi\,. \tag{2.19}$$

The Weyl weights of the superconformal gauge fields follow from the superconformal algebra and are given for $d=6$ in table 3. The Weyl weights of the matter fields can easily be fixed by requiring that the Weyl weight of all terms on the l.h.s. and r.h.s. of each Q transformation rule is equal. For $d=6$ they are given in table 3, together with some further restriction of the fields. All fields are real in the sense explained in appendix A.

(iii) Replace in the Q transformation rules ordinary derivatives ∂_μ by fully covariant derivatives $\hat{\mathscr{D}}_\mu$. Such a derivative is defined as follows. Suppose a field ϕ transforms under the superconformal symmetries with parameters ε^A and gauge fields h_μ^A (cf. (2.5)) as follows:

$$\delta\phi = \varepsilon^A T_A\phi\,, \tag{2.20}$$

where T_A is an operator working on the field ϕ. The fully covariant derivative $\hat{\mathscr{D}}_\mu$ is then defined by

$$\hat{\mathscr{D}}_\mu\phi \equiv \partial_\mu\phi - h_\mu^A T_A\phi\,, \qquad A \neq P_a\,. \tag{2.21}$$

TABLE 3

Fields of $N=2$, $d=6$ conformal supergravity in the formulation with T^-_{abc}

Field	Type	Restrictions	SU(2)	w
e_μ^a	boson	sechsbein	$\underset{\sim}{1}$	-1
ψ_μ^i	fermion	$\gamma_7\psi_\mu^i = +\psi_\mu^i$	$\underset{\sim}{2}$	$-\frac{1}{2}$
V_μ^{ij}	boson	$V_\mu^{ij} = V_\mu^{ji} = (V_{\mu ij})^*(\mu \neq 6)$	$\underset{\sim}{3}$	0
T^-_{abc}	boson	$T^-_{abc} = -\frac{1}{6}\varepsilon_{abcdef}T^-_{def}$	$\underset{\sim}{1}$	1
χ^i	fermion	$\gamma_7\chi^i = \chi^i$	$\underset{\sim}{2}$	$\frac{3}{2}$
D	boson	real	$\underset{\sim}{1}$	2
b_μ	boson	dilatational gauge field	$\underset{\sim}{1}$	0

In other words, the covariantization part is always given by minus the transformation rule of the field with the gauge field h_μ^A substituted for the parameter ε^A. In the following it is understood that whenever the transformation rule of a field is modified, the fully covariant derivative of that field gets also modified according to the above rule. In particular, the superconformal curvatures do get modifications which are either proportional to matter fields or curvature tensors. The latter correspond to modifications in the transformations rules of the type given in (2.13). These modified curvatures are denoted by $\hat{R}$. For the $d=6$ case they are given in table 4.

(iv) Determine the S-supersymmetry transformation rules by requiring that the commutator of a Q-transformation with parameter ε and a K-transformation with parameter Λ_K^a gives a S-transformation, e.g. in $d=6$ (cp. to the $[K, Q] \to S$ commutator given in (B.3)):

$$[\delta_K(\Lambda_K^a), \delta_Q(\varepsilon)] = \delta_S(-\Lambda_K^a \gamma_a \varepsilon) . \tag{2.22}$$

In practice the only field that transforms under K and also occurs in the transformation rules is the dilatation gauge field b_μ. Of course the dependent gauge fields $\omega_\mu{}^{ab}\phi_\mu$ and f_μ^a also transform under K since they depend on b_μ (cf. (2.11)). This fact enables us to give the following rule for deriving the S-transformation rules for all the fields: assemble all the b-dependent terms in the Q-transformation rule (they are hidden in the covariant derivatives). The S-transformation rule (characterized by a parameter η) is then obtained by making in these terms the following substitution (the explicit factors correspond to $d=6$):

$$\tfrac{1}{2}\gamma^\mu b_\mu \varepsilon^i \mapsto \eta^i . \tag{2.23}$$

(v) Replace in the conventional constraints (see (2.10)) the curvatures R by the modified curvatures $\hat{R}$ (see iii.)). Note that the explicit form of these constraints is not important as long as they enable us to solve for the gauge fields $\omega_\mu{}^{ab}$, f_μ^a and ϕ_μ^i. In particular we will use in $d=6$ the following convention. We modify the conventional constraint $\gamma^\mu R_{\mu\nu}(Q)=0$ with the matter field χ in such a way that this constraint is invariant under S-supersymmetry [11]. The advantage of this

TABLE 4

Extension of the superconformal curvatures R to the fully covariant curvatures of $N=2$, $d=6$ conformal supergravity

$$\hat{R}_{\mu\nu}{}^a(P) - R_{\mu\nu}{}^a(P) = 0$$

$$\hat{R}_{\mu\nu}{}^i(Q) - R_{\mu\nu}{}^i(Q) = \tfrac{1}{12}\gamma T^- \gamma_{[\mu}\psi_{\nu]}^i$$

$$\hat{R}_{\mu\nu}{}^{ab}(M) - R_{\mu\nu}{}^{ab}(M) = \bar{\psi}_{[\mu}\gamma^{[a}\hat{R}_{\nu]}{}^{b]}(Q) + \tfrac{1}{2}\bar{\psi}_{[\mu}\gamma_{\nu]}\hat{R}^{ab}(Q) - \tfrac{1}{6}e_{[\mu}{}^{[a}\bar{\psi}_{\nu]}\gamma^{b]}\chi - \tfrac{1}{2}\bar{\psi}_\mu\gamma_c\psi_\nu T^{-abc}$$

$$\hat{R}_{\mu\nu}(D) - R_{\mu\nu}(D) = \tfrac{1}{12}\bar{\psi}_{[\mu}\gamma_{\nu]}\chi$$

$$\hat{R}_{\mu\nu}{}^{ij}(V) - R_{\mu\nu}{}^{ij}(V) = \tfrac{2}{3}\bar{\psi}_{[\mu}^{(i}\gamma_{\nu]}\chi^{j)}$$

$$\hat{R}_{\mu\nu}{}^i(S) - R_{\mu\nu}{}^i(S) = -\tfrac{1}{16}(\gamma^{ab}\gamma_{[\mu} - \tfrac{1}{2}\gamma_{[\mu}\gamma^{ab})\hat{R}_{ab}{}^{ij}(V)\psi_{\nu]j} - \tfrac{1}{48}(\not{\mathcal{D}}\gamma\cdot T^-)\gamma_{[\mu}\psi_{\nu]}^i - \tfrac{1}{24}\bar{\psi}_\mu\gamma^a\psi_\nu\gamma_a\chi^i$$

convention is that the S-transformation of ϕ_μ does not get extra modifications of the type given in (2.13). Of course the Q-transformations of $\omega_\mu{}^{ab}$, f^a_μ and ϕ_μ do get modifications. They can be determined by requiring that under Q-supersymmetry the conventional constraints transform into each other. As a convention we modify in $d=6$ the constraint $R_\mu{}^{ab}(M)e^\nu_b=0$ with additional matter fields in order to simplify the Q-transformation of ϕ_μ. (They eliminate terms of the same type in $\delta_Q\phi_\mu$.) For the explicit form of the $d=6$ conventional constraints, see eq. (2.24) below.

(vi) Construct the full nonlinear Q-transformation rules of all fields. This can be done in the following way. Remember that the linearized transformation rules were determined by requiring that a $[Q, Q]$ commutator gives a translation. However one consequence of the matter fields is that this algebra gets modified. For instance, calculating the $[Q, Q]$ commutator on the sechsbein gives besides a translation also a T^-_{abc} dependent Lorentz transformation (for the full nonlinear $d=6$ superconformal commutator algebra see eq. (2.8) below). In the nonlinear case we now require that the $[Q, Q]$ commutator on all the other fields also gives this T^-_{abc} dependent Lorentz transformation. In general this requires the addition of terms of second order in the fields to the transformation rules. This is the first step of an iterative procedure. In the next step one repeats the calculation on the basis of the new transformation rules, etc. Note that the same iterative procedure as outlined above can also be applied using closure of $[Q, S]$ commutators instead of $[Q, Q]$ commutators. In practice this is much easier since in most cases the $[Q, S]$ commutator algebra does not get matter field modification (cf. to (2.28) below).

This concludes our description of how to construct the full nonlinear Weyl multiplet starting from the linearized transformation rules of the superconformal gauge-fields. We now present our results for the $N=2$, $d=6$ Weyl multiplet. First of all we choose the following set of conventional constraints:

$$
\begin{aligned}
&R_{\mu\nu}{}^a(P)=0\,,\\
&\hat{R}_{\mu\nu}{}^{ab}(M)e^\nu{}_b+T^-_{\mu bc}T^{-abc}+\tfrac{1}{12}e^a_\mu D=0\,,\\
&\gamma^\mu\hat{R}_{\mu\nu}{}^i(Q)=-\tfrac{1}{6}\gamma_\nu\chi^i\,.
\end{aligned}
\tag{2.24}
$$

The definition of the modified curvature $\hat{R}$ is explained in iii. In table 4 we have given the explicit form for most of the curvatures. The solution of (2.24) now includes terms proportional to matter fields (cf. to (2.11)). The solution for $\omega_\mu{}^{ab}$ remains the same (see (2.11)). Those for f^a_μ and ϕ^i_μ are given by:

$$
\begin{aligned}
&f^a_\mu=-\tfrac{1}{8}(\hat{R}'^a_\mu(M)-\tfrac{1}{10}e^a_\mu\hat{R}'(M))+\tfrac{1}{8}T^-_{\mu cd}T^{-acd}+\tfrac{1}{240}e^a_\mu D\,,\\
&\phi^i_\mu=-\tfrac{1}{16}(\gamma^{ab}\gamma_\mu-\tfrac{3}{5}\gamma_\mu\gamma^{ab})\hat{R}'_{ab}{}^i(Q)-\tfrac{1}{60}\gamma_\mu\chi^i\,,\\
&\hat{R}'^a_\mu\equiv\hat{R}'_{\mu\nu}{}^{ba}e^\nu{}_b\,,\qquad \hat{R}'\equiv\hat{R}'_\mu{}^a e^\mu{}_a\,.
\end{aligned}
\tag{2.25}
$$

The notation $\hat{R}'(M)$ and $\hat{R}'(Q)$ indicates that we have omitted the f^a_μ dependent term in $\hat{R}(M)$ and the ϕ_μ dependent terms in $\hat{R}(Q)$ respectively.

We find that the full nonlinear Q and S-transformations of $N=2$, $d=6$ conformal supergravity are given by:

$$
\begin{aligned}
\delta e_\mu^a &= \tfrac{1}{2}\bar\varepsilon\gamma^a\psi_\mu\,,\\
\delta\psi_\mu^i &= \mathcal{D}_\mu\varepsilon^i + \tfrac{1}{24}\,\gamma\cdot T^-\,\gamma_\mu\varepsilon^i + \gamma_\mu\eta^i\,,\\
\delta V_\mu^{ij} &= -4\bar\varepsilon^{(i}\phi_\mu^{j)} - \tfrac{1}{3}\bar\varepsilon^{(i}\gamma_\mu\chi^{j)} - 4\bar\eta^{(i}\psi_\mu{}^{j)}\,,\\
\delta T^-_{abc} &= -\tfrac{1}{32}\bar\varepsilon\gamma^{de}\gamma_{abc}\hat R_{de}(Q) - \tfrac{7}{96}\bar\varepsilon\gamma_{abc}\chi\,,\\
\delta\chi^i &= \tfrac{1}{8}(\hat{\mathcal{D}}_\mu\gamma\cdot T^-)\gamma^\mu\varepsilon^i + \tfrac{3}{16}\gamma\cdot\hat R^{ij}(V)\varepsilon_j + \tfrac{1}{4}D\varepsilon^i + \tfrac{1}{2}\gamma\cdot T\bar\eta^i\,,\\
\delta D &= \bar\varepsilon\gamma^\mu\hat{\mathcal{D}}_\mu\chi - 2\bar\eta\chi\,.
\end{aligned}
\tag{2.26}
$$

Here and in the following we use a shorthand notation for γ-contractions: $\gamma\cdot T^-\equiv\gamma^{abc}T^-_{abc}$, etc. For convenience we also list the Q and S-transformation rules of b_μ and the dependent gauge fields $\omega_\mu{}^{ab}$ and ϕ_μ^i:

$$
\begin{aligned}
\delta b_\mu &= -\tfrac{1}{2}\bar\varepsilon\phi_\mu - \tfrac{1}{24}\bar\varepsilon\gamma_\mu\chi + \tfrac{1}{2}\bar\eta\psi_\mu\,,\\
\delta\omega_\mu{}^{ab} &= -\tfrac{1}{2}\bar\varepsilon\gamma^{ab}\phi_\mu - \tfrac{1}{2}\bar\varepsilon\gamma^{[a}\hat R_\mu{}^{b]}(Q) - \tfrac{1}{4}\bar\varepsilon\gamma_\mu\hat R^{ab}(Q)\\
&\quad -\tfrac{1}{12}e_\mu{}^{[a}\bar\varepsilon\gamma^{b]}\chi + \tfrac{1}{2}\bar\varepsilon\gamma_c\psi_\mu T^{-abc} - \tfrac{1}{2}\bar\eta\gamma^{ab}\psi_\mu\,,\\
\delta\phi_\mu^i &= -f_\mu^a\gamma_a\varepsilon^i + \tfrac{1}{32}(\gamma^{ab}\gamma_\mu - \tfrac{1}{2}\gamma_\mu\gamma^{ab})\hat R_{ab}{}^i{}_j(V)\varepsilon^j\\
&\quad -\tfrac{1}{96}(\hat{\not{\mathcal{D}}}\gamma\cdot T^-)\gamma_\mu\varepsilon^i + \tfrac{1}{24}\gamma_a\chi^i\bar\varepsilon\gamma^a\psi_\mu + \mathcal{D}_\mu\eta^i\,.
\end{aligned}
\tag{2.27}
$$

The algebra of Q and S-transformations in (2.26) and (2.27) closes modulo field-dependent transformations. The algebra takes on the following form:

$$
\begin{aligned}
[\delta_Q(\varepsilon_1),\delta_Q(\varepsilon_2)] &= \tfrac{1}{2}\bar\varepsilon_2\gamma^\mu\varepsilon_1\hat{\mathcal{D}}_\mu + \delta_M(\tfrac{1}{2}\bar\varepsilon_2\gamma_c\varepsilon_1 T^{-abc})\\
&\quad + \delta_S(\tfrac{1}{24}\bar\varepsilon_2\gamma^a\varepsilon_1\gamma_a\chi^i) + \delta_K(\tfrac{1}{8}\bar\varepsilon_2\gamma_b\varepsilon_1\hat{\mathcal{D}}_c T^{-abc} + \tfrac{1}{96}\bar\varepsilon_2\gamma^a\varepsilon_1 D)\\
[\delta_S(\eta),\delta_Q(\varepsilon)] &= \delta_D(-\tfrac{1}{2}\bar\varepsilon\eta) + \delta_M(-\tfrac{1}{2}\bar\varepsilon\gamma^{ab}\eta) + \delta_{\mathrm{SU(2)}}(-4\bar\varepsilon^{(i}\eta^{j)})\\
[\delta_S(\eta_1),\delta_S(\eta_2)] &= \delta_K(-\tfrac{1}{2}\bar\eta_2\gamma^a\eta_1)\,.
\end{aligned}
\tag{2.28}
$$

In the derivation of (2.26)-(2.28) one needs the Q and S variation of $\hat R(Q)$ and $\hat R(V)$. For the convenience of the reader we give these explicitly:

$$
\begin{aligned}
\delta\hat R_{\mu\nu}{}^i(Q) &= \tfrac{1}{16}(-2\gamma_{\mu\nu}{}^{ab} + \gamma_{\mu\nu}\gamma^{ab} + \tfrac{3}{2}\gamma^{ab}\gamma_{\mu\nu})\hat R_{ab}{}^i{}_j(V)\varepsilon^j\\
&\quad + \tfrac{1}{4}(\hat{\mathcal{D}}_{[\mu}T^-_{\nu]ab}\gamma^{ab} - \hat{\not{\mathcal{D}}}\gamma^a T^-_{\mu\nu a})\varepsilon^i + \tfrac{1}{4}\hat R_{\mu\nu}{}^{ab}(M)\gamma_{ab}\varepsilon^i\\
&\quad -\tfrac{1}{4}\gamma_{ab}T^{-abc}T^-_{\mu\nu c}\varepsilon^i + \tfrac{1}{2}T^-_{ab[\mu}\gamma_{\nu]}{}^{ab}\eta^i\,,\\
\delta\hat R_{\mu\nu}{}^{ij}(V) &= -4\bar\varepsilon^{(i}\hat R_{\mu\nu}{}^{j)}(S) + \tfrac{2}{3}\bar\varepsilon^{(i}\gamma_{[\mu}\hat{\mathcal{D}}_{\nu]}\chi^{j)} + \tfrac{1}{3}\bar\varepsilon^{(i}\gamma_a\chi^{j)}T^-_{\mu\nu a}\\
&\quad -4\bar\eta^{(i}\hat R_{\mu\nu}{}^{j)}(Q) - \tfrac{2}{3}\bar\eta^{(i}\gamma_{\mu\nu}\chi^{j)}\,.
\end{aligned}
\tag{2.29}
$$

3. Superconformal matter multiplets

Having constructed the $N=2$, $d=6$ Weyl multiplet in the previous section we now proceed with the matter multiplets. We present here the transformation rules and discuss actions in sect. 4. In this paper we deal with (non-abelian) vector multiplets and scalar multiplets (hypermultiplets) which transform under an optional gauge group associated with the vector fields contained in the vector multiplets. Furthermore we consider the tensor multiplet and both the linear and nonlinear multiplets but not the relaxed hypermultiplet [14]. These multiplets are the building blocks in the construction of general matter couplings to $N=2$, $d=6$ Poincaré supergravity.

3.1. VECTOR MULTIPLETS

The $N=2$, $d=6$ vector multiplet consists of a real vector field W_μ, an SU(2)-Majorana spinor Ω^i of positive chirality (i.e. $\gamma_7\Omega^i=+\Omega^i$) and a triplet of auxiliary scalar fields, $Y^{ij}=(Y_{ij})^*$. In order to derive the (nonlinear) supersymmetry transformation rules of these fields we apply a procedure which very much resembles the one which led us to the construction of the $N=2$, $d=6$ Weyl multiplet in the sect. 2. The same method will be applied to the remaining matter multiplets as well. Therefore we first give a brief resumé of this procedure and then can continue with the vector multiplet. The method consists of the following steps:

(i) First write down the linearized (i.e. only containing terms linear in the fields) Q transformation rules. The explicit values of the coefficients in these transformation rules can be found by requiring that the commutator of two supersymmetries gives a translation (cf. to (2.18)).

(ii) Determine the bosonic transformation rules of all components of the multiplet. In particular assign them a Weyl weight w. For $d=6$ we have listed these Weyl weights for all matter multiplets in table 5, together with some further restrictions on the fields as well.

(iii) Replace in the Q-transformation rules ordinary derivatives ∂_μ by fully covariant derivatives $\hat{\mathscr{D}}_\mu$ (cf. eq. (2.21)).

(iv) Determine the S-supersymmetry transformation rules. As explained in sect. 2 this is most easily done by assembling all b_μ dependent terms in the Q-transformations rules and by making the substitution $\frac{1}{2}\not{b}\,\varepsilon^i\to\eta^i$ (cf. eq. (2.23)).

(v) Construct the full nonlinear Q-transformation rules of all fields by requiring that the superconformal commutator algebra given in eq. (2.28) is realized on all components. Especially the $[Q,S]$ commutator is easily calculated.

This concludes our description of how to construct the full nonlinear transformation rules for all matter multiplets. We now present our results for the $N=2$, $d=6$ vector multiplets. In the non-abelian case all components (W_μ, Ω^i, Y^{ij}) take values in the Lie algebra of the corresponding gauge group. We find that the full nonlinear Q and S-transformation rules are given by:

$$\delta W_\mu=-\bar{\varepsilon}\gamma_\mu\Omega\,,\qquad \delta\Omega^i=\tfrac{1}{8}\gamma\cdot\hat{F}(W)\varepsilon^i-\tfrac{1}{2}Y^{ij}\varepsilon_j$$
$$\delta Y^{ij}=-\bar{\varepsilon}^{(i}\hat{\not{\mathscr{D}}}\Omega^{j)}+2\bar{\eta}^{(i}\Omega^{j)}\,. \tag{3.1}$$

TABLE 5

Fields of various $d=6$ superconformal multiplets: the coefficient w denotes the Weyl weight

Field	Type	Restrictions	SU(2)	w
		Non-abelian vector multiplet		
W_μ	boson	real (non)-abelian gauge field	1	0
Ω^i	fermion	$\gamma_7\Omega^i=+\Omega^i$	2	$\frac{3}{2}$
Y^{ij}	boson	$Y^{ij}=Y^{ji}=Y^*_{ij}$	3	2
		Scalar multiplet		
A^i_α	boson	$A^i_\alpha=\varepsilon^{ij}A^\beta_j\rho_{\beta\alpha}=-(A^\alpha_i)^*$, see eq. (3.3-4)	2	2
ζ^α	fermion	$\gamma_7\zeta^\alpha=-\zeta^\alpha$	1	$\frac{5}{2}$
		Tensor multiplet		
$B_{\mu\nu}$	boson	real antisymmetric tensor gauge field	1	0
ψ^i	fermion	$\gamma_7\psi^i=-\psi^i$	2	$\frac{5}{2}$
σ	boson	real	1	2
		Linear multiplet		
L^{ij}	boson	$L^{ij}=L^{ji}=L^*_{ij}$	3	4
ϕ^i	fermion	$\gamma_7\phi^i=-\phi^i$	2	$\frac{9}{2}$
E_a	boson	real constrained vector; cf. eq. (3.38)	1	5
		Nonlinear multiplet		
Φ^i_α	boson	element of SU(2), $\Phi^i_\alpha=-(\Phi^\alpha_i)^*$	2	0
λ^i	fermion	$\gamma_7\lambda^i=-\lambda^i$	2	$\frac{1}{2}$
V_a	boson	real constrained vector; cf. eq. (3.47)	1	1

The supercovariant field strength $\hat{F}_{\mu\nu}(W)$ of W_μ and the supercovariant derivative $\hat{\mathscr{D}}_\mu\Omega^i$ of Ω^i reads as follows:

$$\begin{aligned}
\hat{F}_{\mu\nu}(W)&=F_{\mu\nu}(W)+2\bar{\psi}_{[\mu}\gamma_{\nu]}\Omega\,,\\
\hat{\mathscr{D}}_\mu\Omega^i&=\partial_\mu\Omega^i-\tfrac{3}{2}b_\mu\Omega^i+\tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\Omega^i-\tfrac{1}{2}V_\mu{}^i{}_j\Omega^j\\
&\quad-\tfrac{1}{8}\gamma\cdot\hat{F}(W)\psi_\mu{}^i+\tfrac{1}{2}Y^{ij}\psi_{\mu j}-gW_\mu\Omega^i\,,\\
\delta\hat{F}_{ab}(W)&=2\bar{\varepsilon}\gamma_{[a}\hat{\mathscr{D}}_{b]}\Omega+\bar{\varepsilon}\gamma^c\Omega T^-_{abc}-2\bar{\eta}\gamma_{ab}\Omega\,.
\end{aligned}\tag{3.2}$$

The fields are Lie-algebra valued, so the last term of $\hat{\mathscr{D}}_\mu\Omega^i$ implies a commutator between the generators. We introduced here and in the following a coupling constant g. For non-simple groups this can be trivially extended to several independent coupling constants.

Note that in $d=6$, unlike to $d=4$, the nonabelian gauge transformation of W_μ does not occur as a field-dependent central charge transformation in the commutator algebra [19].

3.2. SCALAR MULTIPLET

The (on-shell) $N=2$, $d=6$ scalar multiplet (hypermultiplet) consists of scalars A^i_α ($i=1,2$; $\alpha=1\cdots 2r$) which form a doublet of SU(2) and spinors ζ^α which are negative chiral (i.e. $\gamma_7\zeta^\alpha=-\zeta^\alpha$). The index α is a raised or lowered with a matrix $\rho_{\alpha\beta}$

$$A^i_\alpha = A^{i\beta}\rho_{\beta\alpha}\,,\qquad \rho^{\beta\gamma}\rho_{\beta\alpha}=\delta^\gamma_\alpha\,. \tag{3.3}$$

The reality conditions are

$$A^{i\alpha}=A^*_{i\alpha}\,,\qquad A^i_\alpha=-A^{\alpha *}_i\,,\qquad \rho^*_{\alpha\beta}=-\rho^{\beta\alpha}\,. \tag{3.4}$$

The scalar multiplet can transform in a representation of a group which respects these conditions

$$\begin{aligned}&\delta_g A^i_\alpha = t_\alpha{}^\beta A^i_\beta\,,\qquad \delta_g A^{\alpha i}=t^{\alpha\beta}A^i_\beta\,,\\ &t^{\alpha\beta}\equiv\rho^{\alpha\gamma}t_\gamma{}^\beta=(t_\alpha{}^\gamma)^*\rho^{\gamma\beta}\,.\end{aligned} \tag{3.5}$$

The same conditions appeared in 4 dimensions [12] and it was shown that by field definitions of A^i_α the matrix $\rho_{\alpha\beta}$ can be brought in the standard form

$$\rho_{\alpha\beta}=\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots\end{pmatrix}. \tag{3.6}$$

The transformations in (3.5) are by definition those of GL(r, Q), and in fact the fields A^i_α can be interpreted as r quaternions.

The Q and S-transformations can easily be found by applying the procedure outlined in subsect. 3.1. They read as follows

$$\delta A^i_\alpha=\bar\varepsilon^i\zeta_\alpha\,,\qquad \delta\zeta^\alpha=-\tfrac12\hat{\not{\mathscr{D}}}A^\alpha_j\varepsilon^j+2A^\alpha_j\eta^j\,. \tag{3.7}$$

Here the covariant derivative can be gauge invariant with respect to GL(r, Q) or any subgroup

$$\hat{\mathscr{D}}_\mu A^\alpha_i=\partial_\mu A^\alpha_i-2b_\mu A^\alpha_i+\tfrac12 V_{\mu i}{}^jA^\alpha_j-gW_\mu{}^\alpha{}_\beta A^\beta_i-\bar\psi_{\mu i}\zeta^\alpha\,, \tag{3.8}$$

where g is the gauge coupling constant. The superconformal commutator algebra (2.28) is not realised on ζ^α

$$[\delta_Q(\varepsilon_1),\delta_Q(\varepsilon_2)]\zeta^\alpha=\text{eq.(2.28)}-\tfrac14\gamma_\mu\Gamma^\alpha\bar\varepsilon_2\gamma^\mu\varepsilon_1\,. \tag{3.9}$$

Γ^α should therefore be a field equation to have this algebra realised on-shell. The variation of Γ^α under Q-supersymmetry leads then to the field equation for A^i_α. These equations are given by

$$\begin{aligned}&\Gamma^\alpha\equiv\hat{\not{\mathscr{D}}}\zeta^\alpha-\tfrac13A^\alpha_j\chi^j+\tfrac1{12}\gamma\cdot T^-\zeta^\alpha-2g\Omega^{i\alpha}{}_\beta A^\beta_i\\ &C^\alpha_i\equiv(\hat{\mathscr{D}}^a\hat{\mathscr{D}}_a+\tfrac16D)A^\alpha_i+\tfrac16\bar\zeta^\alpha\chi_i+4g\Omega_i{}^\alpha{}_\beta\zeta^\beta+2gY_{ij}{}^\alpha{}_\beta A^{\beta j}\,.\end{aligned} \tag{3.10}$$

These equations are complete in the sense that supersymmetry transformations do not lead to new field equations. Explicitly one finds

$$\begin{aligned}
\delta\Gamma^\alpha &= -\tfrac{1}{2}C_i^\alpha\varepsilon^i - \tfrac{1}{4}\gamma^\mu\gamma^a\Gamma^\alpha\bar{\varepsilon}^i\gamma_a\psi_{\mu i}\,,\\
\delta C_i^\alpha &= \bar{\varepsilon}_i\hat{\mathscr{D}}\Gamma^\alpha + \tfrac{1}{4}\bar{\psi}_{\mu i}\gamma^a\psi_j^\mu\bar{\varepsilon}^j\gamma_a\Gamma^\alpha + 2\bar{\eta}_i\Gamma^\alpha\,,\\
\hat{\mathscr{D}}_\mu\Gamma^\alpha &= \partial_\mu\Gamma^\alpha - \tfrac{7}{2}b_\mu\Gamma^\alpha + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\Gamma^\alpha - gW_\mu{}^\alpha{}_\beta\Gamma^\beta\\
&\quad + \tfrac{1}{2}C_i^\alpha\psi_\mu^i + \tfrac{1}{8}\gamma^\nu\gamma^a\Gamma^\alpha\bar{\psi}_\mu\gamma_a\psi_\nu\,.
\end{aligned} \tag{3.11}$$

The ψ_μ dependent terms in (3.11) can be obtained by requiring that the $\{Q, S\}$ commutator given in (2.28) is realised on Γ^α and C_i^α although these quantities do not form a multiplet. One can also derive the ψ_μ terms in (3.11) directly from the non-closure function (3.9). For this we need a modification of the "theorem on covariant derivatives" in ref. [9] for the case that the algebra does not close. First we define a covariant quantity as one on which no transformation has a derivative on a parameter, hence

$$\delta\phi = \varepsilon^A T_A\phi\,. \tag{3.12}$$

First we suppose that also $T_A\phi$ is a covariant quantity

$$\delta T_A\phi = \varepsilon^B T_B T_A\phi\,, \tag{3.13}$$

but the algebra does not necessarily close on ϕ

$$[\delta(\varepsilon_1), \delta(\varepsilon_2)]\phi = \varepsilon_2^B\varepsilon_1^A[T_A, T_B\}\phi = \varepsilon_2^B\varepsilon_1^A(f_{AB}{}^{\bar{C}}T_{\bar{C}}\phi + \eta_{AB})\,. \tag{3.14}$$

The position and order of indices are written consistently with the conventions for spinor indices as given in appendix A. The notation $\bar{C}$ indicates that the sum over C includes translations. Unbarred indices run over any other transformation. η_{AB} is the "non-closure function on ϕ". We will now consider the transformation of the following covariant derivative

$$\hat{\mathscr{D}}_\mu\phi = \partial_\mu\phi - h_\mu^A T_A\phi\,. \tag{3.15}$$

To calculate $\delta\hat{\mathscr{D}}_\mu\phi$ we use

$$\delta h_\mu^A = \partial_\mu\varepsilon^A + \varepsilon^C h_\mu^{\bar{B}} f_{\bar{B}C}{}^A + \delta_m h_\mu^A\,, \tag{3.16}$$

where $\delta_m h_\mu^A$ are terms containing "covariant" matter fields but not explicit gauge fields (apart from $e_\mu{}^m$). Observe that the ψ_μ matter (T, χ) parts in (2.27) are included in the second term of (3.16) using for $f_{BC}{}^A$ the structure functions of (2.28). We obtain then

$$\begin{aligned}
\delta\hat{\mathscr{D}}_\mu\phi &= \varepsilon^A\partial_\mu T_A\phi - \varepsilon^C h_\mu^{\bar{B}} f_{\bar{B}C}{}^A T_A\phi - h_\mu^A\varepsilon^B T_B T_A\phi - (\delta_m h_\mu^A)T_A\phi\\
&= \varepsilon^A(\partial_\mu - h_\mu^B T_B)T_A\phi - \varepsilon^C e_\mu^a f_{aC}{}^A T_A\phi\\
&\quad - h_\mu^A\varepsilon^B f_{BA}{}^a\mathscr{D}_a\phi - h_\mu^A\varepsilon^B\eta_{BA} - (\delta_m h_\mu^A)T_A\phi\,.
\end{aligned} \tag{3.17}$$

The first term gives the covariant derivative on $T_A\phi$. The second and third term are the remnants of a difference $f_{BC}{}^{\bar{A}} - f_{\bar{B}C}{}^{A}$. Considering now the transformation of

$$\hat{\mathscr{D}}_a\phi = e^\mu_a \hat{\mathscr{D}}_\mu\phi\,, \tag{3.18}$$

we use

$$\delta e^\mu_a = -e^\mu_b e^\nu_a \delta e^b_\nu = -\varepsilon^C h^{\bar{B}}_a f_{\bar{B}C}{}^b e^\mu_b\,. \tag{3.19}$$

Then the third term in (3.17) is cancelled and we obtain

$$\delta\hat{\mathscr{D}}_a\phi = \varepsilon^A\hat{\mathscr{D}}_a T_A\phi + \varepsilon^A h^B_\mu \eta_{BA} - \varepsilon^C f_{aC}{}^{\bar{A}} T_{\bar{A}}\phi - (\delta_{\mathrm{m}} h^A_a) T_A\phi\,. \tag{3.20}$$

The second line of (3.20) are the "covariant terms" in the transformation of the gauge fields in (3.15), e.g. for ψ_μ:

$$\delta\psi_\mu = \tfrac{1}{24}\gamma\cdot T^-\gamma_\mu\varepsilon + \gamma_a e^a_\mu\eta\,. \tag{3.21}$$

The result is thus that if a covariant field transforms only in covariant fields (i.e. (3.12), (3.13)) then the transformation of the covariant derivative with Lorentz index $\hat{\mathscr{D}}_a$ transforms in covariant quantities except for the second term originating from the non-closure function. A well-known consequence of this theorem is that a covariant box is as in (3.10) ($\Box^c A = \hat{\mathscr{D}}^a\hat{\mathscr{D}}_a A = e^{\mu a}\hat{\mathscr{D}}_\mu(e^\nu_a\hat{\mathscr{D}}_\nu A)$).

The second term in $\delta\Gamma^\alpha$ follows from (3.20) and (3.9). The non-closure function on $\hat{\mathscr{D}}_a A^\alpha_i$ follows from the last term of (3.8) and the second term in (3.11) is then immediately obtained. For the definition of $\hat{\mathscr{D}}_\mu\Gamma^\alpha$ we need one more observation. As Γ^α (or in general $\hat{\mathscr{D}}_a\phi$) does transform in a non-covariant term we cannot define $\hat{\mathscr{D}}_a\Gamma^\alpha$ or $\hat{\mathscr{D}}_b\hat{\mathscr{D}}_a\phi$ such that its transformation gives no derivative on a parameter. As $\hat{\mathscr{D}}^a\hat{\mathscr{D}}_a\phi$ contains no covariantization from the noncovariant term in (3.20) ($h^A_a h^B_a\eta_{BA} = 0$) there should be no difficulty in $\delta C^\alpha_i \simeq \delta(\hat{\mathscr{D}}^a\hat{\mathscr{D}}_a A) \simeq \varepsilon\hat{\mathscr{D}}^a\hat{\mathscr{D}}_a\zeta$. In (3.11) occurs

$$\hat{\not{\mathscr{D}}}\hat{\not{\mathscr{D}}}\zeta = \hat{\mathscr{D}}^a\hat{\mathscr{D}}_a\zeta + \tfrac{1}{2}\gamma^{ab}[\hat{\mathscr{D}}_a, \hat{\mathscr{D}}_b]\zeta\,. \tag{3.22}$$

We can define in general $\hat{\mathscr{D}}_b\hat{\mathscr{D}}_a\phi$ such that the antisymmetric part in $[ab]$ does not transform in a derivative on ε

$$\begin{aligned}\hat{\mathscr{D}}_b\hat{\mathscr{D}}_a\phi = \partial_b\hat{\mathscr{D}}_a\phi - h^A_b\hat{\mathscr{D}}_a T_A\phi - \tfrac{1}{2}h^A_b h^B_a\eta_{BA}\\ + h^B_b f_{aC}{}^{\bar{A}} T_{\bar{A}}\phi + (\delta_{\mathrm{m}}(h_b)h^A_a) T_A\phi\,.\end{aligned} \tag{3.23}$$

This definition, which differs from (3.15) by the factor $\frac{1}{2}$ in the third term, satisfies

$$[\hat{\mathscr{D}}_b, \hat{\mathscr{D}}_a]\phi = \hat{R}_{ab}{}^{\bar{A}} T_{\bar{A}}\phi \tag{3.24}$$

and this is the way in which the second term in (3.22) comes in (3.11).

3.3. THE TENSOR MULTIPLET

In rigid superspace [14] one can introduce a real scalar superfield σ satisfying $D^{ij}_{\alpha\beta}\sigma = 0$. The components are a real scalar σ, an SU(2)-Majorana spinor ψ^i of negative chirality (i.e. $\gamma_7\psi^i = -\psi^i$) and a selfdual 3 index tensor F^+_{abc}. They have to be singlets under the gauge group. The full transformations under Q and S are

$$\begin{aligned}
\delta\sigma &= \bar{\varepsilon}\psi\,,\\
\delta\psi^i &= \tfrac{1}{48}\gamma\cdot F^+\varepsilon^i + \tfrac{1}{4}\hat{\not{D}}\sigma\varepsilon^i - \sigma\eta^i\,,\\
\delta F^+_{abc} &= -\tfrac{1}{2}\bar{\varepsilon}\hat{\not{\mathscr{D}}}\gamma_{abc}\psi - 3\bar{\eta}\gamma_{abc}\psi\,,\\
\hat{\mathscr{D}}_\mu\sigma &= (\partial_\mu - 2b_\mu)\sigma - \bar{\psi}_\mu\psi\\
\hat{\mathscr{D}}_\mu\psi^i &= (\partial_\mu - \tfrac{5}{2}b_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab})\psi^i - \tfrac{1}{2}V_\mu{}^i{}_j\psi^j\\
&\quad - \tfrac{1}{48}\gamma\cdot F^+\psi^i_\mu - \tfrac{1}{4}(\hat{\not{\mathscr{D}}}\sigma)\psi^i_\mu + \sigma\phi^i_\mu\,.
\end{aligned} \tag{3.25}$$

The algebra does not close on these fields

$$\begin{aligned}
[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)] &= \text{eq.}(2.28) + \delta_{\text{n.c.}}\,, \qquad \delta_{\text{n.c.}}\sigma = 0\,,\\
\delta_{\text{n.c.}}\psi^i &= \tfrac{1}{8}\gamma^a\Gamma^i\bar{\varepsilon}_1\gamma_a\varepsilon_2\,,\\
\delta_{\text{n.c.}}F^+_{abc} &= -\tfrac{3}{2}\bar{\varepsilon}_2\gamma_{[a}\varepsilon_1 G_{bc]} + \tfrac{3}{4}\bar{\varepsilon}_2\gamma_{[a}\varepsilon_1\bar{\psi}_b\gamma_{c]} + \Gamma\,,\\
\Gamma &= \hat{\not{\mathscr{D}}}\psi - \tfrac{1}{6}\sigma\chi - \tfrac{1}{12}\gamma\cdot T^-\psi\,,\\
G_{ab} &= \hat{\mathscr{D}}^c(F^+_{abc} - 2\sigma T^-_{abc}) - \hat{\hat{R}}_{ab}(Q)\psi - \tfrac{1}{6}\bar{\chi}\gamma_{ab}\psi\,.
\end{aligned} \tag{3.26}$$

The quantities Γ^i and G_{ab} transform among themselves and in an independent quantity C

$$C = (\hat{\mathscr{D}}^a\hat{\mathscr{D}}_a - \tfrac{1}{6}D)\sigma + \tfrac{1}{3}F^+\cdot T^- + \tfrac{7}{6}\bar{\chi}\psi\,. \tag{3.27}$$

These transformations are

$$\begin{aligned}
\delta\Gamma^i &= \tfrac{1}{4}C\varepsilon^i + \tfrac{1}{8}G_{ab}\gamma^{ab}\varepsilon^i - \tfrac{1}{8}\gamma_\mu\gamma_a\Gamma^i\bar{\varepsilon}\gamma_a\psi_\mu\,,\\
\delta G_{ab} &= -\tfrac{1}{2}\bar{\varepsilon}\gamma_{abc}\hat{\mathscr{D}}^c\Gamma + \tfrac{3}{2}\bar{\psi}^c\gamma_{[a}\psi_b\gamma_{c]}\Gamma - \tfrac{1}{2}\bar{\varepsilon}\gamma^c\Gamma T^-_{abc} - 2\bar{\eta}\gamma_{ab}\Gamma\,,\\
\delta C &= \bar{\varepsilon}\hat{\not{\mathscr{D}}}\Gamma + \tfrac{1}{8}\bar{\psi}_\mu\gamma_\nu\Gamma\bar{\varepsilon}\gamma^\nu\psi^\mu + 2\bar{\eta}\Gamma\,.
\end{aligned} \tag{3.28}$$

To obtain closure one should impose the constraints

$$\Gamma^i = G_{ab} = C = 0\,. \tag{3.29}$$

In the domain $\sigma \neq 0$ the first constraint can be used to define χ as a function of fields of the tensor multiplet. The third constraint (using also

$$\begin{aligned}
\hat{\mathscr{D}}^a\hat{\mathscr{D}}_a\sigma &= \partial^a\hat{\mathscr{D}}_a\sigma - 3b^a\hat{\mathscr{D}}_a\sigma + \omega_a{}^{ab}\hat{\mathscr{D}}_b\sigma - 4f_a{}^a\sigma\\
&\quad - \bar{\psi}^\mu\hat{\mathscr{D}}_\mu\psi - \tfrac{1}{12}\sigma\bar{\psi}^\mu\gamma_\mu\chi + \tfrac{1}{24}\bar{\psi}\gamma\cdot T^-\gamma^\mu\psi_\mu + \bar{\psi}\gamma^\mu\phi_\mu
\end{aligned} \tag{3.30}$$

and (2.25)) can be solved for D. The constraint $G_{ab} = 0$ can be solved as a Bianchi identity

$$F^+_{\mu\nu\rho} + 2\sigma T^-_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]} + 3\bar{\psi}_{[\mu}\gamma_{\nu\rho]}\psi + \tfrac{3}{2}\bar{\psi}_{[\mu}\gamma_\nu\psi_{\rho]}\sigma\,. \tag{3.31}$$

$B_{\mu\nu}$ is a newly introduced antisymmetric tensor gauge field which transforms as

$$\delta B_{\mu\nu} = -\bar{\varepsilon}\gamma_{\mu\nu}\psi - \bar{\varepsilon}\gamma_{[\mu}\psi_{\nu]}\sigma + 2\partial_{[\mu}\Lambda_{\nu]} , \tag{3.32}$$

where Λ_μ denotes the gauge invariance of $B_{\mu\nu}$. In fact this modifies (2.28) by a term

$$[\delta(\varepsilon_1), \delta(\varepsilon_2)] = \text{e.q.}(2.28) + \delta_\Lambda(\tfrac{1}{2}\bar{\varepsilon}_2\gamma_\nu\varepsilon_1\sigma) , \tag{3.33}$$

which is a "central charge-like" term. In rigid supersymmetry its field strength is selfdual which prevents the construction of an action [15]. Here T^-_{abc} provides the antiselfdual part, and we will be able to obtain an action of supergravity coupled to a tensor multiplet [1].

At this point we could interpret the system (40+40 Weyl multiplet; 11+8 tensor multiplet; 11+8 constraints) as a new 40+40 Weyl multiplet with independent components

$$(e^a_\mu, \psi^i_\mu, b_\mu, V^{ij}_\mu, B_{\mu\nu}, \psi^i, \sigma) . \tag{3.34}$$

We give the properties of these fields and their transformation rules in appendix C. (See also table 8.)

We also tried a different approach, where we started from $B_{\mu\nu}$, ψ and σ as independent fields as well, but we would not impose (3.29) as constraints, but using field equations. In other words we would construct an on-shell tensor multiplet

$$e^{-1}\mathscr{L} = \sigma C - 4\bar{\psi}\Gamma + \bar{\psi}_\mu\gamma^\mu\sigma\Gamma - \tfrac{1}{2}\partial_{[\mu}B_{\nu\rho]}(F^-_{\mu\nu\rho} - 2\sigma T^-_{\mu\nu\rho}) \tag{3.35}$$

(the left-hand side of (3.31) would then just be called $F^+ + F^-$). However this failed as (3.35) turns out to be non-invariant in order ψ_μ e.g.

$$\delta\mathscr{L} = -\tfrac{3}{2}\bar{\varepsilon}\gamma^{(a}\psi^{d)}F^-_{dbc}F^-_{abc} + \cdots . \tag{3.36}$$

This observation has already been made in [20]. However in the group manifold approach this does not prevent a consistent description.

3.4. THE LINEAR MULTIPLET

The $N=2$, $d=6$ linear multiplet consists of a triplet L^{ij}, an SU(2)-Majorana spinor ϕ^i of negative chirality ($\gamma_7\phi^i = -\phi^i$) and a constrained vector field E_a (see eq. (3.38)). This constraint can be solved in terms of an antisymmetric tensor gauge field $E_{\mu\nu\rho\sigma}$ if the linear multiplet is inert under gauge transformations. The full Q and S transformation rules are given by

$$\begin{aligned}
\delta L^{ij} &= \bar{\varepsilon}^{(i}\varphi^{j)} ,\\
\delta\varphi^i &= \tfrac{1}{2}\hat{\not{\mathscr{D}}}L^{ij}\varepsilon_j - \tfrac{1}{4}\gamma^a E_a\varepsilon^i - 4L^{ij}\eta_j ,\\
\delta E_a &= \bar{\varepsilon}\gamma_{ab}\hat{\mathscr{D}}^b\varphi + \tfrac{1}{24}\bar{\varepsilon}\gamma_a\gamma\cdot T^-\varphi - \tfrac{1}{3}\bar{\varepsilon}^i\gamma_a\chi^j L_{ij} - 5\bar{\eta}\gamma_a\varphi - 2g\bar{\varepsilon}_i\gamma_a\Omega_j L^{ij} ,\\
\hat{\mathscr{D}}_\mu L^{ij} &= (\partial_\mu - 4b_\mu)L^{ij} - V_\mu{}^{(i}{}_k L^{j)k} - \bar{\psi}_\mu{}^{(i}\varphi^{j)} - gW_\mu L^{ij} ,\\
\hat{\mathscr{D}}_\mu\varphi^i &= (\partial_\mu - \tfrac{9}{2}b_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab})\varphi^i + \tfrac{1}{2}V^{ij}_\mu\varphi_j - gW_\mu\varphi^i\\
&\quad - \tfrac{1}{2}\hat{\not{\mathscr{D}}}L^{ij}\psi_{\mu j} + \tfrac{1}{4}\gamma^a E_a\psi^i_\mu + 4L^{ij}\phi_{\mu j} ,
\end{aligned} \tag{3.37}$$

where W_μ and Ω stand for $W_\mu{}^A t_A$, $\Omega^A t_A$ and t_A are the representation matrices. The algebra closes if E_a satisfies the following Q- and S-invariant constraint

$$\begin{aligned}
&\hat{\mathscr{D}}^a E_a - \tfrac{1}{2}\bar{\varphi}\chi - 4g\bar{\Omega}\varphi - 2gY^{ij}L_{ij} = 0\,,\\
&\hat{\mathscr{D}}_\mu E_a \equiv (\partial_\mu - 5b_\mu)E_a + \omega_{\mu ab}E^b - gW_\mu E_a - \bar{\psi}_\mu \gamma_{ab}\hat{\mathscr{D}}^b\varphi\\
&\qquad - \tfrac{1}{24}\bar{\psi}_\mu\gamma_a\gamma\cdot T^-\varphi + \tfrac{1}{3}\bar{\psi}^i_\mu\gamma_a\chi^j L_{ij} + 5\bar{\phi}_\mu\gamma_a\varphi + 2g\bar{\psi}_{\mu i}\gamma_a\Omega_j L^{ij}\,.
\end{aligned}\tag{3.38}$$

For $g=0$ the solution for E_a in terms of $E_{\mu\nu\rho\sigma}$ is

$$E^a = \tfrac{1}{24} i e^{-1} e^a_\mu \varepsilon^{\mu\nu\rho\sigma\lambda\tau}\hat{\mathscr{D}}_\nu E_{\rho\sigma\lambda\tau}\tag{3.39}$$

and there is a gauge invariance

$$\delta_\Lambda E_{\mu\nu\rho\sigma} = 4\partial_{[\mu}\Lambda_{\nu\rho\sigma]}\,,\tag{3.40}$$

or for the dual 2-index field we have

$$\begin{aligned}
E_{\mu\nu\rho\sigma} &= -\tfrac{1}{2} i e \varepsilon_{\mu\nu\rho\sigma\lambda\tau}E^{\lambda\tau}\,,\\
E^a &= e^a_\mu \hat{\mathscr{D}}_\nu E^{\mu\nu}\,,\\
\delta E^{\mu\nu} &= \bar{\varepsilon}\gamma^{\mu\nu}\varphi + \bar{\psi}^i_\rho\gamma^{\mu\nu\rho}\varepsilon^j L_{ij} + \partial_\rho(e^{-1}\tilde{\Lambda}^{\mu\nu\rho})\,,\\
\hat{\mathscr{D}}_\nu E^{\mu\nu} &= \partial_\nu E^{\mu\nu} - \bar{\psi}_\nu\gamma^{\mu\nu}\varphi - \tfrac{1}{2}\bar{\psi}^i_\rho\gamma^{\mu\nu\rho}\psi^j_\nu L_{ij}\,.
\end{aligned}\tag{3.41}$$

Also here there is a "central charge-like" term in the algebra (see (3.33))

$$[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)] = \text{e.q.}(2.28) + \delta_\Lambda(\bar{\varepsilon}^i_2\gamma^{\mu\nu\rho}\varepsilon^j_1 L_{ij})\,.\tag{3.42}$$

So it is the antiselfdual part of the tensor gauge transformation which appears in $\{Q, Q\}$.

3.5. THE NONLINEAR MULTIPLET

All matter multiplets discussed so far have transformation rules that are linear in the components of the multiplet itself. The nonlinear part of these transformations is entirely caused by the presence of the fields of the $N=2$, $d=6$ Weyl multiplet. We will now discuss a multiplet which does not possess this property. The nonlinear transformations are induced by the presence of a bosonic constraint of the type encountered in nonlinear sigma models. The multiplet contains Lorentz scalar fields $\Phi^i_\alpha(\alpha=1,2)$ of zero Weyl weight which parametrize an element of SU(2). Hence the 2×2 matrix Φ^i_α satisfies

$$\Phi^i_\alpha\Phi^\alpha_j = \delta^i_j\,,\qquad \Phi^\alpha_i\Phi^i_\beta = \delta^\alpha_\beta\,.\tag{3.43}$$

The indices $\alpha, \beta, \ldots$ are raised and lowered by means of the invariant tensors $\varepsilon^{\alpha\beta}$ and $\varepsilon_{\alpha\beta}$ according to

$$\Phi^i_\alpha = \varepsilon^{ij}\Phi^\beta_j\varepsilon_{\beta\alpha}\,.\tag{3.44}$$

From (3.23) it follows that Φ^i_α represents three spin-zero degrees of freedom. Under supersymmetry Φ^i_α transforms into an SU(2)-Majorana spinor λ^i of negative chirality (i.e. $\lambda^i = -\gamma_7\lambda^i$) and a real Lorentz vector field V_a.

This vector field turns out to have the noteworthy feature that it transforms under special conformal transformations (K), as it does in $d=4$ [11]. The full Q, S and K transformations of the multiplet read

$$\begin{aligned}
\delta\Phi^i_\alpha &= \bar\varepsilon^{(i}\lambda^{j)}\Phi_{j\alpha}\,,\\
\delta\lambda^i &= \tfrac{1}{2}\Phi^i_\alpha\hat{\not{\mathscr{D}}}\Phi^\alpha_j\varepsilon^j - \tfrac{1}{4}\gamma^a V_a\varepsilon^i + \tfrac{1}{4}\gamma_a\varepsilon_j\bar\lambda^i\gamma^a\lambda^j\\
&\quad + \tfrac{1}{96}\gamma_{abc}\varepsilon^i\bar\lambda^k\gamma^{abc}\lambda_k - 2\eta^i\,,\\
\delta V_a &= \bar\varepsilon\gamma_{ab}\hat{\mathscr{D}}^b\lambda + \tfrac{1}{6}\bar\varepsilon\gamma_a\chi - \tfrac{1}{2}\bar\varepsilon\gamma_a\gamma^b\lambda V_b + \tfrac{1}{24}\bar\varepsilon\gamma_a\gamma\cdot T^-\lambda\\
&\quad - \bar\varepsilon_i\gamma_a\Phi^i_\alpha\hat{\not{\mathscr{D}}}\Phi^\alpha_j\lambda^j + 2g\bar\varepsilon^i\gamma_a(\Omega_j)^\alpha{}_\beta\Phi_\alpha{}^j\Phi^\beta_i - \bar\eta\gamma_a\lambda - 8\Lambda_{Ka}\,.
\end{aligned}\tag{3.45}$$

Here we included the possibility of gauging the SU(2) or an SO(2) subgroup corresponding to the index α by means of a vector multiplet (W_μ, Ω, Y_{ij}). The covariant derivatives $\hat{\mathscr{D}}_\mu\Phi^i_\alpha$ and $\hat{\mathscr{D}}_\mu\lambda^i$ are given by

$$\begin{aligned}
\hat{\mathscr{D}}_\mu\Phi^i_\alpha &= \partial_\mu\Phi^i_\alpha + \tfrac{1}{2}V^{ij}_\mu\Phi_{j\alpha} - gW_{\mu\alpha}{}^\beta\Phi^i_\beta - \bar\psi^{(i}_\mu\lambda^{j)}\Phi_{j\alpha}\,,\\
\hat{\mathscr{D}}_\mu\lambda^i &= (\partial_\mu - \tfrac{1}{2}b_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab})\lambda^i - \tfrac{1}{2}V_\mu{}^i{}_j\lambda^j - \tfrac{1}{2}\Phi^i_\alpha\hat{\not{\mathscr{D}}}\Phi^\alpha_j\psi^j_\mu\\
&\quad + \tfrac{1}{4}\gamma^a V_a\psi^i_\mu - \tfrac{1}{4}\gamma_a\psi_{\mu j}\bar\lambda^i\gamma^a\lambda^j - \tfrac{1}{96}\gamma_{abc}\psi^i_\mu\bar\lambda^k\gamma^{abc}\lambda_k + 2\phi^i_\mu\,.
\end{aligned}\tag{3.46}$$

The superconformal algebra (2.28) closes if and only if we have the following Q, S and K invariant constraint of Weyl weight 2

$$\begin{aligned}
&\mathscr{D}^a V_a - \tfrac{1}{3}D - \tfrac{1}{2}V_aV^a + (\hat{\mathscr{D}}^a\Phi^i_\alpha)\hat{\mathscr{D}}_a\Phi^\alpha_i + 2\bar\lambda\hat{\not{\mathscr{D}}}\lambda + \tfrac{5}{6}\bar\lambda\chi + \tfrac{1}{6}\bar\lambda\gamma\cdot T^-\lambda\\
&\quad + 2\bar\lambda^i\Phi_{\alpha i}\hat{\not{\mathscr{D}}}\Phi^\alpha_j\lambda^j + 2g(Y_{ij})^\alpha{}_\beta\Phi^i_\alpha\Phi^{\beta j} + 8g\bar\lambda^i(\Omega_j)^\alpha{}_\beta\Phi^j_\alpha\Phi^\beta_i = 0\,,
\end{aligned}\tag{3.47}$$

where the supercovariant derivative $\hat{\mathscr{D}}_\mu V_a$ is given by

$$\begin{aligned}
\hat{\mathscr{D}}_\mu V_a &= \partial_\mu V_a - b_\mu V_a + \omega_{\mu a}{}^b V_b + 8f^a_\mu - \bar\psi_\mu\gamma_{ab}\hat{\mathscr{D}}^b\lambda\\
&\quad - \tfrac{1}{6}\bar\psi_\mu\gamma_a\chi + \tfrac{1}{2}\bar\psi_\mu\gamma_a\gamma^b\lambda V_b - \tfrac{1}{24}\bar\psi_\mu\gamma_a\gamma\cdot T^-\lambda\\
&\quad + \bar\psi_{\mu i}\gamma_a\Phi^i_\alpha\hat{\not{\mathscr{D}}}\Phi^\alpha_j\lambda^j + \bar\phi_\mu\gamma_a\lambda - 2g\bar\psi^i_\mu\gamma_a(\Omega_j)^\alpha{}_\beta\phi^j_\alpha\Phi^\beta_i\,.
\end{aligned}\tag{3.48}$$

4. Actions

In this section we will obtain actions for matter multiplets coupled to Poincaré supergravity. We will start with the action for the scalar multiplets. It will turn out that this action formula applied to a compensating scalar multiplet gives the action for Poincaré supergravity coupled to a tensor multiplet. The second action formula which we need is that for a product of a linear multiplet with a vector multiplet. This formula can then be used in two ways. The linear multiplet can be the "kinetic multiplet" (or multiplet of currents) of the vector multiplet: i.e. its components are functions of the components of the vector multiplet. In this way we obtain the kinetic terms for the vector multiplets. The other way is exactly as in four dimensions [21, 22], a vector multiplet can be defined from the components of a tensor multiplet. Inserted in the action formula this produces a superconformal invariant action for the tensor multiplet ("improved linear multiplet"). This multiplet can also be used as a compensating multiplet. One obtains then an off-shell formulation of Poincaré supergravity coupled to a tensor multiplet (see table 6).

4.1. SCALAR MULTIPLET ACTION

As we know the field equations from subsect. 3.2 we can easily write an invariant action similar to the one known in 4 dimensions [19, 21].

$$e^{-1}\mathscr{L}_S = A_i^\alpha d_\alpha{}^\beta C^i_\beta + (2\bar{\zeta}^\alpha + \bar{\psi}^i_\mu \gamma^\mu A_i^\alpha) d_\alpha{}^\beta \Gamma_\beta \,. \tag{4.1}$$

We introduced here a tensor $d_\alpha{}^\beta$ with the following properties

$$d_{\alpha\beta} = -d_{\beta\alpha}\,, \qquad d_{\alpha\beta} \equiv -d_\alpha{}^\gamma \rho_{\gamma\beta}\,,$$
$$(d_\alpha{}^\beta)^* = d_\beta{}^\alpha\,. \tag{4.2}$$

The first line can be assumed as the symmetric part of $d_{\alpha\beta}$ would give a total divergence in (4.1). The second equation makes (4.1) real. The action is gauge invariant for transformations (3.5) which satisfy

$$(t_\gamma{}^\alpha)^* d_\gamma{}^\beta + d_\alpha{}^\gamma t_\gamma{}^\beta = 0\,. \tag{4.3}$$

TABLE 6

Independent fields of the off-shell formulation of Poincaré supergravity coupled to the tensor multiplet

		Number of components
e^a_μ	physical graviton field, gauges general coordinate transformations	15
ψ^i_μ	physical gravitino; gauge field of supersymmetry	−40
V^{ij}_a	auxiliary SU(2) triplet vector	2×6
	$V_{\mu ij}\delta^{ij}$ gauges an SO(2) invariance	+5
σ	physical scalar. Singularity at $\sigma = 0$	1
ψ^i	physical fermions	−8
$B_{\mu\nu}$	physical antisymmetric tensor gauge field	10
$E_{\mu\nu\rho\sigma}$	auxiliary antisymmetric tensor with 3-index gauge invariance	5

In [12] it was shown that field redefinitions can simultaneously bring $\rho_{\alpha\beta}$ in the canonical form (3.6) and $d_\alpha{}^\beta$ in the form

$$d_\alpha{}^\beta = \begin{pmatrix} \mathbb{1}_p & \\ & -\mathbb{1}_q \end{pmatrix} \qquad (p, q \text{ even})\,. \tag{4.4}$$

For physical fields we have $d_\alpha{}^\beta = -\delta_\alpha{}^\beta$, but as we will explain shortly we also need "compensating fields" for which $d_\alpha{}^\beta = \delta_\alpha{}^\beta$. Transformations which satisfy (3.5) and (4.3) are those of USp(p, q).

To get a better understanding of the content of (4.1) we write explicitly the bosonic terms. Using (3.10) and (2.25) we obtain after a partial integration

$$\begin{aligned} e^{-1}\mathscr{L}_{\mathrm{S,B}} = &-\partial_a A_i^\alpha d_\alpha{}^\beta \partial_a A_\beta{}^i + \tfrac{1}{2}(A_j^\alpha \overleftrightarrow{\partial}_a A_\beta{}^i) V_{ai}{}^j d_\alpha{}^\beta \\ &+ \tfrac{1}{8} V_{ai}{}^j V_{aj}{}^i A^2 + \tfrac{1}{5} R A^2 + \tfrac{1}{15} D A^2\,, \end{aligned} \tag{4.5}$$

where

$$A^2 = A_i{}^\alpha d_\alpha{}^\beta A_\beta{}^i \tag{4.6}$$

and we write R for $\hat{R}$ in (2.25). At this point we can replace D by the components of the tensor multiplet. The constraint (3.29) gives for the bosonic part

$$\tfrac{4}{15}\sigma D = \mathscr{D}^a \mathscr{D}_a \sigma + \tfrac{1}{5}\sigma R + \tfrac{1}{3} F^+ \cdot T^-\,. \tag{4.7}$$

We then obtain

$$\begin{aligned} e^{-1}\mathscr{L}_{\mathrm{S,B}} = &\tfrac{1}{4} R A^2 - \partial_a A_i^\alpha d_\alpha{}^\beta \partial_a A_\beta^i + \tfrac{1}{2}(A_j^\alpha \overleftrightarrow{\partial}_a A_\beta^i) V_{ai}{}^j d_\alpha{}^\beta \\ &+ \tfrac{1}{8} V_{ai}{}^j V_{aj}{}^i A^2 + \tfrac{1}{4} A^2 \sigma^{-1} \mathscr{D}^a \mathscr{D}_a \sigma + \tfrac{1}{12} A^2 \sigma^{-1} F^+ \cdot T^-\,. \end{aligned} \tag{4.8}$$

We can use then as dilatational gauge choice

$$D\text{-gauge:} \qquad A^2 = -2 \tag{4.9}$$

to have the standard Einstein gravity action as first term in (4.8). Then we can use the field equation of $V_{ai}{}^j$ to get

$$\begin{aligned} e^{-1}\mathscr{L}_{\mathrm{S,B}} = &-\tfrac{1}{2} R - \partial_a A_i^\alpha (d_\alpha{}^\beta + d_\alpha{}^\gamma A_\gamma^j A_j^\delta d_\delta{}^\beta) \partial_a A_\beta{}^i \\ &- \tfrac{1}{2}\sigma^{-2} \partial_a \sigma \partial_a \sigma - \tfrac{3}{4}\sigma^{-2} \partial^{[\mu} B^{\nu\rho]} \partial_\mu B_{\nu\rho}\,. \end{aligned} \tag{4.10}$$

The action thus contains automatically Poincaré supergravity and a tensor multiplet (observe that the fields have positive kinetic energy). To be able to impose (4.9) we need at least one multiplet (two values of $\alpha, \beta \ldots$) with the + sign in (4.4) (remember (3.4) to see this). This multiplet is the "compensating multiplet". One can say that (4.9) fixes one of the four scalar fields of this multiplet. The other 3 are fixed by an SU(2) gauge choice e.g.

$$\text{SU(2) gauge:} \qquad A_i^\alpha \sim \delta_i^\alpha \qquad \text{for } \alpha = 1, 2\,. \tag{4.11}$$

The interactions of the scalar multiplet correspond to fields taking values on the quaternionic projective space

$$\frac{\mathrm{USp}(2,2n)}{\mathrm{USp}(2)\times\mathrm{USp}(2n)}\,. \tag{4.12}$$

We expect that the other quaternionic manifolds mentioned in [23] are contained in our action (4.1) by considering multiplets transforming in a gauge group but without kinetic terms for the vector multiplets. The field equations for the fields of the vector multiplet change the manifold and we use more compensating scalar multiplets to fix gauges of the gauge group. All this is completely the same as in 4 dimensions, where this procedure was first applied to construct the manifold [24]

$$\frac{\mathrm{SU}(2,n)}{\mathrm{SU}(2)\times\mathrm{SU}(n)\times\mathrm{U}(1)}\,. \tag{4.13}$$

4.2. ACTION FORMULA FOR PRODUCT OF VECTOR AND LINEAR MULTIPLET

The action formula we are looking for is contained in the constraint for the linear multiplet (3.38) which has Weyl weight 6. In fact if we select there the g-dependent terms we obtain the following density

$$\begin{aligned} e^{-1}\mathscr{L}_{\mathrm{VL}} = {} & Y_{ij}L^{ij} + 2\bar{\Omega}\varphi - L^{ij}\bar{\psi}_{\mu i}\gamma^{\mu}\Omega_j \\ & + \tfrac{1}{2}W_a\big(E^a - \bar{\psi}_b\gamma^{ba}\varphi - \tfrac{1}{2}\bar{\psi}_{bi}\gamma^{abc}\psi_{cj}L^{ij}\big)\,. \end{aligned} \tag{4.14}$$

In (3.38) the fields of the vector multiplet are Lie algebra valued: however, in (4.14) we consider an abelian vector multiplet, and the fields are not supposed to be multiplied by representation matrices acting on the linear multiplet components. The linear multiplet can transform under the abelian group with an arbitrary weight. One can convince oneself from the gauge invariance as the transformation $\delta W_a = \partial_a \Lambda$ gives after partial integration the g independent terms of (3.38) while by construction of (4.14) the transformation of the linear multiplet components gives the g-dependent terms of (3.38). If the linear multiplet is inert under the gauge group, then (3.41) can be used to rewrite the second line of (4.14)

$$e^{-1}\mathscr{L}_{\mathrm{VL}} = Y_{ij}L^{ij} + 2\bar{\Omega}\varphi - L^{ij}\bar{\psi}_{\mu i}\gamma^{\mu}\Omega_j + \tfrac{1}{4}F_{\mu\nu}(W)E^{\mu\nu}e^{-1}\,. \tag{4.15}$$

4.3. VECTOR MULTIPLET ACTION

In four dimensions the action of the vector multiplet is conformally invariant, essentially because

$$(\partial_\mu W_\nu - \partial_\nu W_\mu)(\partial_\rho W_\sigma - \partial_\sigma W_\rho)g^{\mu\rho}g^{\nu\sigma} \tag{4.16}$$

has Weyl weight four. In 6 dimensions this should be multiplied by a quantity of Weyl weight 2 in order to be a candidate for a conformal invariant action. In other words we want to construct a linear multiplet of field equations for the vector multiplet. In 4 dimensions this could start from $L^{ij} = Y^{ij}$ but here there is a mismatch of Weyl weights and we need compensator scalar fields. One could think of several possibilities: scalars belonging to scalar multiplets, linear multiplets or the tensor multiplet or even a combination of them. In any case it is not difficult to write an S-invariant candidate for L^{ij}. To define a linear multiplet the Q-transformation of L^{ij} should be as in (3.37) i.e. only an SU(2) doublet. This corresponds to the superspace constraints

$$\mathscr{D}_\alpha{}^{(k}L^{ij)} = 0\,. \tag{4.17}$$

This singles out the expression

$$L^{ij} = \sigma Y^{ij} + 2\bar{\psi}^{(i}\Omega^{j)}\,. \tag{4.18}$$

So it is the tensor multiplet which enters in this construction. As this multiplet anyway enters in the construction of Poincaré supergravity this is a solution which does not raise the number of independent fields. One now straightforwardly computes the other components of the linear multiplet (for an abelian vector multiplet)

$$\begin{aligned}
\varphi^i &= -\sigma\hat{\not{\mathscr{D}}}\Omega^i + Y^{ij}\psi_j + \tfrac{1}{4}\gamma^{ab}\hat{F}_{ab}(W)\psi^i \\
&\quad + \tfrac{1}{24}\gamma^{abc}\hat{F}^+_{abc}(B)\Omega^i - \tfrac{1}{2}(\hat{\not{\mathscr{D}}}\sigma)\Omega^i \\
E_a &= \hat{\mathscr{D}}^b(\hat{F}_{ba}(W)\sigma) + \tfrac{1}{2}(\hat{F}^+(B) - 2\sigma T^-)_{abc}\hat{F}^{bc}(W) \\
&\quad + 2\hat{\mathscr{D}}_b(\bar{\Omega}\gamma^{ab}\psi) + \tfrac{1}{3}\bar{\Omega}\gamma_a\chi\sigma\,.
\end{aligned} \tag{4.19}$$

One can verify (3.38) using (3.29). For the abelian case this constraint can even be solved for the antisymmetric tensor

$$\begin{aligned}
E_{\mu\nu} &= -\sigma F_{\mu\nu}(W) + \tfrac{1}{4}i\varepsilon_{\mu\nu\rho\sigma\lambda\tau}B^{\rho\sigma}F^{\lambda\tau}(W) \\
&\quad + 2\bar{\Omega}\gamma_{\mu\nu}\psi + \sigma\bar{\Omega}\gamma^\rho\gamma_{\mu\nu}\psi_\rho\,.
\end{aligned} \tag{4.20}$$

Inserting (4.18)–(4.20) in the action formula (4.14) or (4.15) one obtains an action for the abelian vector multiplet*

$$\begin{aligned}
e^{-1}\mathscr{L}_{\mathrm{V}} &= \sigma[-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - 2\bar{\Omega}\not{\mathscr{D}}\Omega + Y^{ij}Y_{ij}] \\
&\quad + \tfrac{1}{16}ie^{-1}\varepsilon_{\mu\nu\rho\sigma\lambda\tau}B^{\mu\nu}F^{\rho\sigma}F^{\lambda\tau} - 4\bar{\Omega}^i\psi^jY_{ij} \\
&\quad + \tfrac{1}{4}\bar{\Omega}\gamma^\mu\gamma^{\nu\rho}(F_{\nu\rho} + \hat{F}_{\nu\rho})(\sigma\psi_\mu + \gamma_\mu\psi) + \tfrac{1}{12}\bar{\Omega}\gamma\cdot\hat{F}^+(B)\Omega \\
&\quad + 2\bar{\Omega}^{(i}\psi^{j)}\bar{\psi}_{\mu i}\gamma^\mu\Omega_j\,.
\end{aligned} \tag{4.21}$$

It turns out that (4.21) is also an invariant for nonabelian vector multiplets. Obviously one then writes a trace before the whole expression, and the fields are again Lie algebra valued.

* We remark the striking similarity between this action and the analogous result in 10 dimensions [25, 26].

4.4. ACTION OF THE LINEAR MULTIPLET

We will reverse now the procedure which we used in the previous section. We construct now a vector multiplet as multiplet of field equations for the fields of the linear multiplet. Again we need scalar fields to compensate for a mismatch of the Weyl weights. But this time, as in $N=1$ [21] and $N=2$ [22] in four dimensions, we can use the scalar fields L^{ij} of the multiplet itself as Weyl compensator. This kind of action was called the improved tensor multiplet in [21, 22]. We define as there

$$L=(L^{ij}L_{ij})^{1/2}\,. \tag{4.22}$$

We determine then Ω by S-invariance and by demanding the Q-transformation of the form (3.1), which determines then also $\hat{F}_{\mu\nu}$ and Y^{ij}. The results are

$$\begin{aligned}
\Omega^i &= L^{-1}\hat{\mathscr{D}}\varphi^i - L^{-3}(\hat{\mathscr{D}}L^{ij})L_{jk}\varphi^k+\tfrac{1}{2}L^{-3}\gamma^a E_a L^{ij}\varphi_j+\tfrac{2}{3}L^{-1}L^{ij}\chi_j\\
&\quad+\tfrac{1}{12}L^{-1}\gamma\cdot T^-\varphi^i+\tfrac{1}{2}L^{-5}L^{ij}\gamma_a\varphi_j L^{kl}\bar{\varphi}_k\gamma_a\varphi_l+2gL^{-1}L^{ij}\Omega_j\,,\\
Y^{ij} &= -L^{-1}\hat{\mathscr{D}}_a\hat{\mathscr{D}}^a L^{ij}+L^{-3}L_{kl}\hat{\mathscr{D}}^a L^{k(i}\hat{\mathscr{D}}_a L^{j)l}+\tfrac{1}{4}L^{-3}E_aE^aL^{ij}\\
&\quad-L^{-3}E_aL^{k(i}\hat{\mathscr{D}}_aL^{j)}{}_k-\tfrac{1}{3}L^{-1}L^{ij}D+\tfrac{1}{6}L^{-1}\bar{\chi}^{(i}\varphi^{j)}\\
&\quad-\tfrac{4}{3}L^{-3}L^{k(i}L^{j)l}\bar{\chi}_k\varphi_l+\tfrac{1}{4}L^{-3}L^{ij}\bar{\varphi}^k\hat{\mathscr{D}}\varphi_k+2L^{-3}L^{k(i}\bar{\varphi}_k\hat{\mathscr{D}}\varphi^{j)}\\
&\quad-L^{-3}\hat{\mathscr{D}}_aL^{k(i}\varphi^{j)}\gamma^a\varphi_k-3L^{-5}L^{pq}L^{k(i}\hat{\mathscr{D}}_aL^{j)}{}_k\bar{\varphi}_p\gamma^a\varphi_q\\
&\quad-\tfrac{1}{12}L^{-3}L^{ij}\bar{\varphi}^k\gamma\cdot T^-\varphi_k+\tfrac{1}{4}L^{-3}\bar{\varphi}^{(i}\gamma^aE_a\varphi^{j)}\\
&\quad+\tfrac{3}{2}L^{-5}L^{k(i}L^{j)l}\bar{\varphi}_k\gamma^a\varphi_lE_a-\tfrac{1}{2}L^{-5}\bar{\varphi}^{(i}\gamma_a\varphi^{j)}(L^{kl}\bar{\varphi}_k\gamma_a\varphi_l)\\
&\quad+\tfrac{5}{4}L^{-7}L^{ij}(L^{kl}\bar{\varphi}_k\gamma_a\varphi_l)(L^{mn}\bar{\varphi}_m\gamma^a\varphi_n)+2gL^{-1}Y^{k(i}L^{j)}{}_k\\
&\quad-2gL^{-1}\bar{\Omega}^{(i}\varphi^{j)}-4gL^{-3}L^{k(i}L^{j)l}\bar{\Omega}_k\varphi_l\,,\\
\hat{F}_{ab}(W) &= -L^{-1}L^{ij}\hat{R}_{abij}(V)-2\hat{\mathscr{D}}_{[a}(L^{-1}E_{b]})-2L^{-3}L^l_k\hat{\mathscr{D}}_{[a}L^{kp}\hat{\mathscr{D}}_{b]}L_{lp}\\
&\quad+L^{-1}\bar{\hat{R}}_{ab}{}^k(Q)\varphi_k-2\hat{\mathscr{D}}_{[a}(L^{-3}L^{ij}\bar{\varphi}_i\gamma_{b]}\varphi_j)\,.
\end{aligned}\tag{4.23}$$

$\hat{E}_{ab}$ satisfies a Bianchi identity, but exactly as in 4 dimensions the $L\partial L\partial L$ term cannot be written as a derivative of a gauge vector in an SU(2) covariant way. This cohomology problem is related to the singularity at $L=0$. This has been discussed in detail in [22]. This implies that we cannot use (4.14), but only (4.15) to write down an action. The linear multiplet should thus be gauge inert as we need the antisymmetric tensor. $\hat{F}_{ab}(L)$ is obtained from (3.2) and the bosonic part can be written as

$$F_{\mu\nu}(L)=-2\partial_\mu L^{ij}L^k_j\partial_\nu L^i_kL^{-3}-2\partial_{[\mu}(E_{\nu]}L^{-1}+V_{\nu]ij}L^{ij}L^{-1})\,. \tag{4.24}$$

The bosonic part of the action for the linear multiplet is then

$$\begin{aligned}
e^{-1}\mathscr{L}_{\mathrm{L,B}} &= -\hat{\mathscr{D}}_a\hat{\mathscr{D}}^aL+\tfrac{1}{2}L^{-1}\mathscr{D}_aL_{ij}\mathscr{D}^aL^{ij}-\tfrac{1}{3}LD-\tfrac{1}{4}L^{-1}E^aE_a\\
&\quad-\tfrac{1}{2}E^\mu V_{\mu ij}L^{ij}L^{-1}-\tfrac{1}{2}E^{\mu\nu}\partial_\mu L^{ij}L^k_j\partial_\nu L_{ki}\,.
\end{aligned}\tag{4.25}$$

In sect. 3 of [22] it was checked that in 4 dimensions and in the absence of supergravity this action is equivalent to the scalar multiplet action. This has been done using a duality transformation and field redefinitions, and also by checking S-matrix elements.

We will now consider also the application of the linear multiplet as compensator multiplet. Using that $\hat{\mathscr{D}}^a\hat{\mathscr{D}}_a L$ contains $-8f_a{}^a L$, (2.25) and (4.7) we obtain for $\mathscr{L}_{\mathrm{L,B}}$

$$\begin{aligned} e^{-1}\mathscr{L}_{\mathrm{L,B}} = &-\tfrac{1}{2}LR+\tfrac{1}{2}L^{-1}\partial_a L^{ij}\partial^a L_{ij}+L^{-1}V_a^{ij}L_i{}^k\partial_a L_{kj}+\tfrac{1}{2}V_a^{(i}{}_k L^{j)k}V_{ail}L_j^l \\ &-\tfrac{1}{2}L\sigma^{-1}\mathscr{D}^a\mathscr{D}_a\sigma-\tfrac{1}{6}L\sigma^{-1}F^+\cdot T^- -\tfrac{1}{4}L^{-1}E^aE_a \\ &-\tfrac{1}{2}E^\mu V_{\mu ij}L^{ij}L^{-1}-\tfrac{1}{2}E^{\mu\nu}\partial_\mu L^{ij}L_j^k\partial_\nu L_{ki}\,. \end{aligned} \tag{4.26}$$

As dilatational gauge choice we adopt then

$$D\text{-gauge:}\qquad L=1\,. \tag{4.27}$$

(If we have a combined system $\mathscr{L}_{\mathrm{S}}+\mathscr{L}_{\mathrm{L}}$ we would take $L-\frac{1}{2}A^2=1$). SU(2) cannot be completely broken in this case. A gauge choice

$$L^{ij}=\sqrt{\tfrac{1}{2}}\delta^{ij} \tag{4.28}$$

breaks $D\otimes\mathrm{SU}(2)$ to an SO(2) subgroup. This SO(2) is then gauged by the auxiliary field $V_{\mu ij}\delta^{ij}$. The action (4.26) now reduces to

$$\begin{aligned} e^{-1}\mathscr{L}_{\mathrm{L,B}} = &-\tfrac{1}{2}R+\tfrac{1}{8}V_a^{ij}V_{aij}-\tfrac{1}{2}\sigma^{-2}\partial_a\sigma\partial^a\sigma-\tfrac{1}{4}E^aE_a \\ &-\tfrac{1}{2}\sqrt{\tfrac{1}{2}}E^\mu V_{\mu ij}\delta^{ij}-\tfrac{3}{4}\sigma^{-2}\partial^{[\mu}B^{\nu\rho]}\partial_\mu B_{\nu\rho}\,. \end{aligned} \tag{4.29}$$

We obtain, as in subsect. 4.1 an action for Poincaré supergravity coupled to a tensor multiplet. However in this case *we used as compensator multiplet the linear multiplet, which is an off-shell representation. So we obtain here a formulation of the theory with an off-shell closed algebra.* For clarity, we repeat the independent fields in table 6 with the number of off-shell components (negative for fermions). Observe that ϕ^i of the linear multiplet has disappeared due to S-invariance. This formulation has a remaining SO(2) invariance as in the "new minimal" auxiliary field formulation of $N=1$, $d=4$ supergravity [16]. The last term in (4.29) is characteristic of this formulation.

5. Conclusions

In this paper we constructed local supersymmetric matter couplings in six dimensions. Throughout our work we applied superconformal techniques. We first construct a superconformal invariant action. After imposing appropriate gauge conditions these actions lead to matter coupled Poincaré supergravity.

By adding a tensor multiplet to the Weyl multiplet of sect. 2 and imposing constraints necessary for closure of the algebra, we came to another version of the Weyl multiplet. This version resembles the Weyl multiplet in $d=10$ supergravity

[26]. There one has introduced differential constraints on a scalar and a spinor. Here we started with fields D and χ which were determined by these constraints such that the independent fields have no differential constraints.

The fields of this 40+40 multiplet contain the physical fields for supergravity coupled to a tensor multiplet. To obtain a lagrangian we have to add a second compensating multiplet. In $N=2$, $d=4$ we had 3 possibilities at this stage. The first one used a non-linear multiplet [11]. This does not help us here because it does not offer an action (in $d=4$ the action came from the first compensating multiplet and the non-linear multiplet could make the field equation consistent). The second possibility [11] makes use of a scalar multiplet. This is the most useful choice for the construction of hypermultiplet couplings and considerations on their symmetries [12]. We can use that also in $d=6$ as we explained at the end of subsect. 4.1. As the scalar multiplet is not an off-shell multiplet this does not give us an auxiliary field formulation of $N=2$, $d=6$ supergravity. Such a formulation is obtained here only using the third possibility [22] which has a linear multiplet as second compensator. It is an auxiliary field formulation which is comparable with the new minimal form of $N=1$, $d=4$ in the sense that it has a SO(2) invariance and an auxiliary antisymmetric tensor field.

The general action which we obtained can be written as

$$\mathscr{L}=\mathscr{L}_{\mathrm{S}}+\mathscr{L}_{\mathrm{V}}+\mathscr{L}_{\mathrm{L}}+g\mathscr{L}_{\mathrm{LV}}\,.$$

$\mathscr{L}_{\mathrm{S}}$ or $\mathscr{L}_{\mathrm{L}}$ contains a compensating multiplet as discussed above, and the action describes at least supergravity and a tensor multiplet. $\mathscr{L}_{\mathrm{S}}$ can contain an arbitrary number of hypermultiplets and describes their kinetic terms and gauge interactions. We obtain in this sector quaternionic Kähler sigma models [23, 24]. In particular by its field equation the SU(2) gauge field $V_\mu{}^{ij}$ of the Weyl multiplet becomes the composite SU(2) connection of this manifold. $\mathscr{L}_{\mathrm{V}}$ contains the kinetic action for the vector multiplet. These vector multiplets are in $\mathscr{L}_{\mathrm{V}}$ also coupled to the tensor multiplet, e.g. we obtain a term

$$e^{-1}\varepsilon^{\mu\nu\rho\sigma\lambda\tau}B_{\mu\nu}\mathrm{Tr}(F_{\rho\sigma}F_{\lambda\tau})\,.$$

$\mathscr{L}_{\mathrm{L}}$ can describe also physical linear multiplets, probably equivalent to hypermultiplets. $\mathscr{L}_{\mathrm{LV}}$, which we have used previously to construct $\mathscr{L}_{\mathrm{L}}$ and $\mathscr{L}_{\mathrm{V}}$ can also be added as an extra term with independent fields of linear and vector multiplets to describe interactions probably equivalent to U(1) gauge interactions of scalar multiplets. g in (5.1) is a coupling constant.

As an example of the possibilities in (5.1) we mention the gauged $N=2$ supergravities. In the formulation with the compensating scalar multiplet we can gauge an SU(2) or SO(2) acting on the $\alpha, \beta, \ldots$ indices of the compensating multiplet by physical vector multiplets as in (3.5). The gauge condition (4.11) implies then that the Poincaré theory is invariant under a diagonal subgroup of this gauge group and the SU(2) automorphism group of the supersymmetries acting on the $i, j, \ldots$. This invariance is gauged by the vector multiplets. A cosmological term is generated by the Y_{ij} field equation (see the last term of (3.10) inserted in (4.1) and using the nonzero values of the field A_i^α, compensator of dilatations).

In the off-shell formulation (with the linear multiplet compensator) we can gauge only the SO(2) group acting on the gravitinos. This is done by adding $\mathscr{L}_{\mathrm{LV}}$ for the compensator tensor multiplet and a U(1) vector multiplet. The field equation of E_μ then implies

$$V_{\mu ij}\delta^{ij} = gW_\mu .$$

Since $V_{\mu ij}$ couples to the gravitinos, W_μ is the gauge vector of the SO(2) automorphism group after the field equations. Y_{ij} gets again a constant term due to the first terms in (4.15). It implies a cosmological constant. Observe that in $d=6$ the cosmological constant in gauged supergravity is a positive constant in the potential as it originates from Y_{ij}. In fact Y_{ij} getting a constant term is like a Fayet–Iliopoulos term, breaking supersymmetry spontaneously.

Throughout this work we found much the same mechanisms as those in $N=2$, $d=4$ superconformal tensor calculus. However, some new aspects arose especially concerning the tensor multiplet. Apart from the intrinsic value of the construction of 6-dimensional matter couplings in view of compactifications (see the Introduction), this construction can shed some light on the structure of 10-dimensional conformal supergravity.

Appendix A

NOTATIONS AND CONVENTIONS

Throughout this paper we use the Pauli metric $\delta_{\mu\nu} = \mathrm{diag}\,(+, \dots, +)$. The 8×8 Dirac matrices γ_a $(a = 1, \dots, 6)$ are defined by the property

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\delta_{ab} \tag{A.1}$$

and are hermitian. A complete set of 8×8 matrices O_I satisfying tr $O_I O_I = 8\delta_{IJ}$ is given by

$$O_I = \{\mathbb{1}, \gamma^{(1)}, i\gamma^{(2)}, i\gamma^{(3)}, \gamma^{(4)}, \gamma^{(5)}, i\gamma^{(6)}\}, \tag{A.2}$$

where we have used the following notation

$$\gamma^{(n)} = \gamma^{a_1\dots a_n} = \gamma^{[a_1}\gamma^{a_2}\cdots\gamma^{a_n]} = \frac{1}{n!}\sum_p (-1)^p \gamma^{a_1}\cdots\gamma^{a_n}, \tag{A.3}$$

where $\sum_p$ means summation over all permutations. The matrix γ_7 which is defined by

$$\gamma_7 = i\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6 \tag{A.4}$$

allows one to express $\gamma^{(n)}$ in terms of $\gamma^{(6-n)}$. This duality relation reads

$$\gamma^{a_1\ldots a_n} = i\frac{S_n}{(6-n)!}\varepsilon^{a_1\ldots a_n b_1\ldots b_{6-n}}\gamma_{b_1\ldots b_{6-n}}\gamma_7\,, \qquad S_n = \begin{cases} +1: & n=0,1,4,5 \\ -1: & n=2,3,6 \end{cases}. \tag{A.5}$$

The charge conjugation matrix C in six dimensions satisfies [27]

$$C^{\mathrm{T}} = C\,, \qquad C^{\dagger}C = 1\,, \qquad \gamma_a^{\mathrm{T}} = -C\gamma_a C^{-1}\,, \qquad \gamma_7^{\mathrm{T}} = -C\gamma_7 C^{-1}\,. \tag{A.6}$$

Our spinors carry an SU(2) index i ($i = 1, 2$). This index can be raised or lowered by means of the ε symbol ε^{ij}:

$$\lambda^i = \varepsilon^{ij}\lambda_j\,, \qquad \lambda_i = \lambda^j\varepsilon_{ji}\,. \tag{A.7}$$

Spinors in six dimensions can satisfy an SU(2)-Majorana and Weyl condition respectively:

$$\bar\lambda^i \equiv (\lambda_i)^\dagger\gamma_6 = (\lambda^i)^{\mathrm{T}}C\,,$$

where "6" denotes the time component and e.g. for a left-handed spinor

$$\lambda^i = \tfrac{1}{2}(1+\gamma_7)\lambda^i\,, \qquad \bar\lambda^i = \bar\lambda^i\tfrac{1}{2}(1-\gamma_7)\,. \tag{A.8}$$

It follows that $\bar\lambda^i\chi_i$ is a real scalar quantity if c.c. includes the change of order of fermionic quantities*. Tensors $\bar\lambda^i\gamma_a\chi_i$, $\bar\lambda^i\gamma_{ab}\chi_i\ldots$ are real except for timelike components. When SU(2) indices are omitted, a northwest-southeast contraction is understood, e.g.

$$\bar\lambda\gamma^{(n)}\psi = \bar\lambda^i\gamma^{(n)}\psi_i\,. \tag{A.9}$$

Changing the order of spinors in a bilinear leads to the following signs:

$$\bar\lambda\gamma^{(n)}\psi = t_n\bar\psi\gamma^{(n)}\lambda\,, \qquad t_n = \begin{cases} +: & n=0,3,4 \\ -: & n=1,2,5,6 \end{cases} \tag{A.10}$$

An additional sign is needed if the SU(2) indices are not contracted, e.g. $\bar\lambda\gamma^a\psi = -\bar\psi\gamma^a\lambda$ but $\bar\lambda^i\gamma^a\psi j = +\bar\psi j\gamma^a\lambda^i$.

The completeness relation tr $O_IO_J = 8\delta_{IJ}$ (see A.2) leads to the following Fierz rearrangement formulas:

$$\psi_j\bar\lambda^i = -\tfrac{1}{8}\{2(\bar\lambda^i\gamma^a\psi_j)\gamma_a - \tfrac{1}{6}(\bar\lambda^i\gamma^{abc}\psi_j)\gamma_{abc}\}\tfrac{1}{2}(1+\gamma_7)\,, \tag{A.11a}$$

if ψ and λ are both negative chiral and

$$\psi_j\bar\lambda^i = -\tfrac{1}{8}\{2\bar\lambda^i\psi_j - (\bar\lambda^i\gamma^{ab}\psi_j\gamma_{ab}\}\tfrac{1}{2}(1+\gamma_7)\,, \tag{A.11b}$$

if ψ is positive chiral and λ is negative chiral.

* For fermionic operators as in appendix B one should realise that we insert only a sign whereas the order of the operators is not changed: $(Q_{i\alpha}Q_{j\beta})^* = -(\bar Q^i\gamma_6)^\alpha(\bar Q^j\gamma_6)^\beta$.

One of the more frequently occurring calculations with γ-matrices is the product [28]

$$\gamma^{(n)}\gamma^{(m)}\gamma_{(n)} = n!\,C(n,m)\gamma^{(m)}\,. \tag{A.12}$$

The coefficients $C(n, m)$ for $n, m \leq 3$ are given in table 7.

Products of γ-matrices are given by:

$$\begin{aligned}\gamma_{a_1\cdots a_n}\gamma^{b_1\cdots b_m} &= \gamma_{a_1\cdots a_n}{}^{b_1\cdots b_m} + (-1)^{n+1}\binom{n}{1}\binom{m}{1}1!\,\delta^{[b_1}_{[a_1}\gamma_{a_2\cdots a_n]}{}^{b_2\cdots b_m]}\\ &\quad + (-1)^{2n+1}\binom{n}{2}\binom{m}{2}2!\,\delta^{[b_1}_{[a_1}\delta^{b_2}_{a_2}\gamma_{a_3\cdots a_n]}{}^{b_3\cdots b_m]}\\ &\quad + (-1)^{n}\binom{n}{3}\binom{m}{3}3!\,\delta^{[b_1}_{[a_1}\delta^{b_2}_{a_2}\delta^{b_3}_{a_3}\gamma_{a_4\cdots a_n]}{}^{b_4\cdots b_m]} + \cdots\,.\end{aligned} \tag{A.13}$$

Throughout this paper the dual $\tilde{T}_{abc}$ of a tensor T_{abc} is defined by

$$\tilde{T}_{abc} = \tfrac{1}{6} i\varepsilon_{abcdef}T^{def} \tag{A.14}$$

which means that $\tilde{\tilde{T}} = T$. Positive and negative dual parts are defined by:

$$T^{\pm}_{abc} = \tfrac{1}{2}(T_{abc} \pm \tilde{T}_{abc})\,. \tag{A.15}$$

Differently then in $d = 4$ (anti)selfdual tensors can be real in the sense that the component with $abc \neq 6$ are real. We remark that $T^{+}_{abc}T^{-abc} = \frac{1}{2}T_{abc}T^{abc}$ is non-zero but the product of two tensors of the same duality vanishes

$$T^{+}_{abc}T^{+abc} = 0\,. \tag{A.16}$$

Other useful identities for (anti)selfdual tensors are

$$\begin{aligned} &F^{+}_{acd}T^{-bcd} + F^{+bcd}T^{-}_{acd} = \tfrac{1}{3}\delta^{b}_{a}F^{+}_{cde}T^{-cde}\\ &F^{+}_{cd[a}T^{+}_{b]}{}^{cd} = 0\,, \qquad F^{+}_{a[bc}T^{+}_{de]}{}^{a} = 0\,. \end{aligned} \tag{A.17}$$

The 3-γ tensors $(1 \pm \gamma_7)\gamma_{abc}$ are $\pm$selfdual, respectively. Although we usually omit spinor indices we still need them for general formula as (2.1), (2.4), (2.5). The

TABLE 7

The coefficients $C(n, m)$ defined in (A.12)

m \ n	0	1	2	3
0	1	6	−15	−20
1	1	−4	−5	0
2	1	2	1	4
3	1	0	3	0

conventions are then the following. Spinor indices $\alpha, \beta \ldots$ are raised or lowered with the symmetric tensors $C^{\alpha\beta}$ or $C_{\alpha\beta}$ which are each other inverse and complex conjugate

$$C^{\alpha\beta}C_{\beta\gamma} = \delta^{\alpha}_{\gamma}\,, \qquad (C^{\alpha\beta})^* = C_{\alpha\beta}\,. \tag{A.18}$$

The usual γ-matrices have their indices in SW–NE direction, so e.g.

$$\bar{\chi}\gamma_a\lambda = \chi^{\alpha}\gamma_{a\alpha}{}^{\beta}\lambda_{\beta}\,. \tag{A.19}$$

The γ-matrices are antisymmetric in their spinor indices

$$(\gamma_a)_{\alpha\beta} \equiv \gamma_{a\alpha}{}^{\gamma}C_{\gamma\beta} = -\gamma_{a\beta\alpha} \tag{A.20}$$

and satisfy

$$(\gamma_{a\alpha\beta})^* = -\gamma_a{}^{\alpha\beta}\,. \tag{A.21}$$

Finally remark that our (anti)symmetrizations are with weight one

$$[ab] = \tfrac{1}{2}(ab - ba)\,. \tag{A.22}$$

Appendix B

THE SUPERCONFORMAL ALGEBRA IN SIX DIMENSIONS

The 28-parameter conformal group in six dimensions consists of the 21-parameter Poincaré group (translation P_a and rotations M_{ab}), dilatations D and special conformal transformations K_a. The algebra is given by

$$\begin{aligned}
[M_{ab}, M^{cd}] &= -2\delta^{[c}_{[a}M_{b]}{}^{d]}\,,\\
[P_a, M_{bc}] &= P_{[c}\delta_{b]a}\,,\\
[K_a, M_{bc}] &= K_{[c}\delta_{b]a}\,,\\
[P_a, K_b] &= -2(\delta_{ab}D + 2M_{ab})\,,\\
[D, P_a] &= P_a\,,\\
[D, K_a] &= -K_a\,.
\end{aligned} \tag{B.1}$$

We next introduce two supersymmetries Q^i_α $(i = 1, 2)$. The spinor generator Q^i_α are SU(2)-Majorana–Weyl

$$Q^i_\alpha \equiv (Q^\beta_i)^*(\gamma_6)^\beta{}_\alpha = C_{\alpha\beta}\varepsilon^{ij}Q^\beta_j\,, \qquad Q^i_\alpha = \tfrac{1}{2}(1-\gamma_7)_\alpha{}^\beta Q^i_\beta\,. \tag{B.2}$$

The $[K, Q]$ commutator produces two other "special" supersymmetries S^i_α. The spinor generators S^i_α are SU(2)-Majorana–Weyl and positive chiral. The Jacobi identities demand that $[Q^i_\alpha, S^j_\beta]$ contains D, M and moreover an SU(2) triplet generator U_{ij} which satisfies $U_{ij} = U_{ji} = (U^{ij})^*$, $U^i_i = 0$. The new (anti) commutators are

$$
\begin{aligned}
&\{Q^i_\alpha, Q^j_\beta\} = -\tfrac{1}{2}(\gamma_a)_{\alpha\beta}\varepsilon^{ij}P^a\,,\\
&\{S^i_\alpha, S^j_\beta\} = +\tfrac{1}{2}(\gamma_a)_{\dot\alpha\beta}\varepsilon^{ij}K^a\,,\\
&\{Q^i_\alpha, S^j_\beta\} = \tfrac{1}{2}C_{\alpha\beta}\varepsilon^{ij}D + \tfrac{1}{2}(\gamma_{ab})_{\alpha\beta}\varepsilon^{ij}M^{ab} - 4C_{\alpha\beta}U^{ij}\,,\\
&[M_{ab}, Q^i_\alpha] = -\tfrac{1}{4}(\gamma_{ab}Q)^i_\alpha\,,\\
&[M_{ab}, S^i_\alpha] = -\tfrac{1}{4}(\gamma_{ab}S)^i_\alpha\,,\\
&[U_{ij}, Q_{\alpha k}] = \tfrac{1}{2}\varepsilon_{k(i}Q_{j)\alpha}\,, \qquad [U_{ij}, S_{\alpha k}] = \tfrac{1}{2}\varepsilon_{k(i}S_{j)\alpha}\,,\\
&[U_{ij}, U_{kl}] = -\tfrac{1}{2}U_{il}\varepsilon_{jk} + \tfrac{1}{2}U_{kj}\varepsilon_{li}\,,\\
&[D, Q^i_\alpha] = \tfrac{1}{2}Q^i_\alpha\,, \qquad [D, S^i_\alpha] = -\tfrac{1}{2}S^i_\alpha\,,\\
&[K_a, Q^i_\alpha] = (\gamma_a S)^i_\alpha\,, \qquad [P_a, S^i_\alpha] = -(\gamma_a Q)^i_\alpha\,.
\end{aligned}
\tag{B.3}
$$

In (B.3) we have used a shorthand notation to denote chiral projections of Dirac matrices, e.g.

$$
(\gamma_a)_{\dot\alpha\beta} = (\tfrac{1}{2}(1+\gamma_7)\gamma_a)_{\alpha\beta}\,, \qquad \gamma_{\alpha\beta} = (\tfrac{1}{2}(1-\gamma_7)\gamma_a)_{\alpha\beta}
\tag{B.4}
$$

and the same for all other matrices. The superconformal generators satisfy generalized Jacobi identities. As an example, we prove the (Q, Q, S) Jacobi identity. First substituting the commutators (B.3) yields:

$$
\begin{aligned}
&[\{Q^i_\alpha, Q^j_\beta\}, S^k_\gamma] - [Q^i_\alpha, \{Q^j_\beta, S^k_\gamma\}] - [Q^j_\beta, \{Q^i_\alpha, S^k_\gamma\}]\\
&\quad = \tfrac{1}{4}\varepsilon^{ij}(\gamma_a)_{\alpha\beta}(\gamma^a Q)^k_\gamma - \tfrac{1}{8}(\gamma_{ab})_{\beta\gamma}\varepsilon^{jk}(\gamma^{ab}Q)^i_\alpha\\
&\qquad - 2C_{\beta\gamma}(\varepsilon^{ij}Q^k_\alpha + \tfrac{3}{8}\varepsilon^{jk}Q^i_\alpha) + (\alpha i \leftrightarrow \beta j)\,.
\end{aligned}
\tag{B.5}
$$

We next apply the Fierz relation (cf. eq. (A.11a))

$$
M_{\alpha\beta} = \tfrac{1}{8}\{2\,\mathrm{Tr}\,(\gamma_a M)(\gamma^a)_{\alpha\beta} - \tfrac{1}{6}\,\mathrm{Tr}\,(\gamma_{abc}M)(\gamma^{abc})_{\alpha\beta}\}
\tag{B.6}
$$

on the second and third terms at the r.h.s. of (B.5). Then the γ^{abc} terms cancel between each other and the γ^a terms cancel against the first term at the r.h.s. of (B.5). This proves the Jacobi identity $(Q, Q, S) = 0$. Thus we have obtained a parametrization of the supergroup OSp(6, 2/1).

Appendix C

THE WEYL MULTIPLET WITH THE TENSOR GAUGE FIELD $B_{\mu\nu}$

The Weyl multiplet in the formulation with the tensor gauge field $B_{\mu\nu}$ can be obtained from the Weyl multiplet given in sect. 2 (see e.g. eqs. (2.26)–(2.28) by

TABLE 8

Fields of $N=2$, $d=6$ conformal supergravity in the formulation with $B_{\mu\nu}$

Field	Type	Restrictions	SU(2)	w
e_μ^a	boson	sechsbein	1	-1
ψ_μ^i	fermion	$\gamma_7\psi_\mu^i=+\psi_\mu^i$	2	$-\frac{1}{2}$
V_μ^{ij}	boson	$V_\mu^{ij}=V_\mu^{ji}=(V_{\mu ij})^*(\mu\neq 6)$	3	0
$B_{\mu\nu}$	boson	$\delta B_{\mu\nu}=2\partial_{[\mu}\Lambda_{\nu]}$	1	0
ψ^i	fermion	$\gamma_7\psi^i=-\psi^i$	2	$\frac{5}{2}$
σ	boson	real	1	2
b_μ	boson	dilatational gauge field	1	0

substituting for the fields T_{abc}^-, χ^i and D the solution of the constraints (3.31), (3.26) ($\Gamma=0$) and (3.27) respectively. Note that since T_{abc}^- always multiplies a positive dual expression (or an expression which can be brought into that form) F_{abc}^+ always drops out in the substitution for T_{abc}^-. The resulting transformation rules are the following

$$
\begin{aligned}
\delta e_\mu^a &= \tfrac{1}{2}\bar\varepsilon\gamma^a\psi_\mu\,,\\
\delta\psi_\mu^i &= \mathcal{D}_\mu\varepsilon^i+\tfrac{1}{48}\sigma^{-1}\gamma\cdot\hat F(B)\gamma_\mu\varepsilon^i+\gamma_\mu\eta^i\,,\\
\delta B_{\mu\nu} &= -\bar\varepsilon\gamma_{[\mu}\psi_{\nu]}\sigma-\bar\varepsilon\gamma_{\mu\nu}\psi+2\partial_{[\mu}\Lambda_{\nu]}\,,\\
\delta V_\mu^{ij} &= -4\bar\varepsilon^{(i}\phi_\mu^{j)}+\tfrac{1}{12}\sigma^{-2}\bar\varepsilon^{(i}\gamma\cdot\hat F(B)\psi^{j)}-2\sigma^{-1}\bar\varepsilon^{(i}\gamma_\mu\hat{\not{\mathcal{D}}}\psi^{j)}-4\bar\eta^{(i}\psi_r{}^{j)}\,,\\
\delta\psi^i &= \tfrac{1}{48}\gamma\cdot\hat F(B)\varepsilon^i+\tfrac{1}{4}\hat{\not{\mathcal{D}}}\sigma\varepsilon^i-\sigma\eta^i\,,\\
\delta\sigma &= \bar\varepsilon\psi\,.
\end{aligned}
$$

The properties of these fields are given in table 8. We emphasize that all the properties of this multiplet (such as the commutator algebra and curvatures) can be obtained from sect. 2 by making the replacements mentioned above for T_{abc}^-, χ^i and D.

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